Coherent states and the classical limit on irreducible SU_3 representations

1998 J. Phys. A: Math. Gen. 31 9871

(http://iopscience.iop.org/0305-4470/31/49/011)

J. Phys. A: Math. Gen. **31** (1998) 9871–9896. Printed in the UK PII: S0305-4470(98)96116-9

Coherent states and the classical limit on irreducible SU_3 representations

Sven Gnutzmann† and Marek Kuś†‡

† Fachbereich Physik, Universität GH Essen, 45117 Essen, Germany
‡ Center for Theoretical Physics, Polish Academy of Sciences, Warsaw, Poland

Received 21 July 1998

Abstract. We give an explicit construction of the SU_3 coherent states for an arbitrary irreducible representation. We also construct the symplectic structure on the manifold of coherent states, find canonical variables and discuss various classical limits of quantum-mechanical systems with relevant observables that obey $\boldsymbol{su}_3$ commutation relations.

1. Introduction

Coherent states [1], which originated from the early paper of Schrödinger [2] were always thought of as providing a possible link between quantum and classical mechanics. In works of Glauber [3], Klauder [4] and Sudarshan [5] the coherent states of the Heisenberg–Weyl group were constructed as eigenstates of the annihilation operator and applied to study properties (such as, for example, coherence) of the quantized electromagnetic radiation. In fact, in this case, one can characterize the coherent states in three equivalent ways: (i) as the eigenstates of the annihilation operator; (ii) as the states minimizing the Heisenberg uncertainty relations and (iii) as the states obtained by acting on the vacuum state by Heisenberg–Weyl group operators. As was observed by Perelomov [6] the third definition can be extended to an arbitrary Lie group. In fact, for the group SU_2 the corresponding coherent states were described earlier by Radcliffe [7], and soon applied to the problems of atomic physics [8, 9], where the SU_2 group is a natural symmetry group of a system consisting of two-level atoms.

The usefulness of the Perelomov generalized coherent states to the analysis of the classical limit of quantum systems was shown by Yaffe [10]. He considered a very common situation in which a quantum system involves a large number of degrees of freedom N (e.g. consists of N atoms). Often it is possible to define physical quantities (e.g. energy or polarization per one atom) which have finite values in the limit $N \to \infty$. Moreover, the corresponding quantum observables become 'classical' (e.g. in the sense of the uncertainty relations—their mutual commutators formally approach zero) and their dynamical behaviour is governed by classical equations of motion. The classical phase space in which this motion takes place is constructed with the help of appropriate coherent states.

In our paper we want to present explicitly this construction for the case of the SU_3 group. The physical motivation is provided, as mentioned above, by problems of many-atom systems. If only three levels of each atom are of importance (e.g. due to a resonant interaction with light) then SU_3 is the natural symmetry of the system and the number of atoms N is connected to dimensions of representations of SU_3 which are relevant to the

0305-4470/98/499871+26$19.50

problem. This observation was extensively used in the analysis of the so-called superradiant laser [11]. The analogous construction, as well as a lot of useful relations fulfilled by the coherent states in the case of the SU_2 group are well known (see the above cited papers of Radcliffe, Arecchi *et al* and Glauber and Haake). The SU_3 group is much richer than SU_2—its irreducible representations are parametrized by two numbers (in contrast to SU_2 where the total spin uniquely characterizes each representation). In the case of SU_2 the dimension of representation is proportional to the total spin which, in the case of systems of two-level atoms, corresponds to the number of atoms N. There is only one way of attaining the classical limit $N \to \infty$ corresponding to increasing to infinity the dimensionality of representations. In the case of SU_3 the same limit (the number of atoms N tends to infinity) may be realized in different ways in terms of representations. In fact, the actual limit may depend, for example, on our assumptions about the initial preparation of the quantum states of the system and can lead to different properties of the classical limiting dynamics. For the superradiant laser this situation was analysed thoroughly in [11].

There is also another interesting aspect of the above-mentioned ambiguity in the definition of the classical limit for the quantum systems with the SU_3 symmetry. It is believed that some spectral properties of quantum systems are determined by their classical behaviour [12]. The statistical distributions of energy levels are different for systems which are classically chaotic and integrable. The problem becomes interesting if we realize that in the case of SU_3 we can construct two different classical dynamics from one quantum Hamiltonian—we will elaborate the topic elsewhere [13].

Despite the fact that the general construction of the coherent states for the SU_3 (and, in fact, SU_N [14–17]) group is known, we were not able to find in the available literature the explicit construction of the classical phase space, canonical variables, etc. The same applies to, closely connected, explicit formulae for the expectation values of various generators of SU_3 in the coherent state representation—usually the authors concentrate on one type of representation only.

This paper is organized as follows. In section 2 we describe briefly some relevant facts concerning the group SU_3 and its representations, in section 3 we give the definition and discuss various properties of the SU_3 coherent states. It is this section which contains formulae useful in quantum-mechanical calculations concerning expectation values. The last subsection of section 3 recalls the geometric meaning of the coherent states. Section 4 is devoted to the natural symplectic structure of the manifold of the coherent states and its usefulness in defining the classical limit of quantum systems with SU_3 symmetry. In particular, in section 4.2 we give explicitly a coordinate form of the Poisson brackets (as well as of the symplectic form). The canonical coordinates are given explicitly in section 4.4.

2. The Lie groups SU_3 and $SL_3(\mathbb{C})$

2.1. Definition of SU_3 and $SL_3(\mathbb{C})$

The Lie group SU_3 is defined as the set of unitary 3×3 matrices with determinant equal to one and with matrix multiplication as the group operation. It is embedded in its complexification $SL_3(\mathbb{C})$, the group of arbitrary complex 3×3 matrices with determinant equal to one. Physically one may think of SU_3 as the set of all basis transformations of a three-level system (regarding two bases as equal if they differ by a global phase factor).

Any group element of SU_3 and $SL_3(\mathbb{C})$ acts as an automorphism in $\mathbb{C}^3$ with its standard basis

$$|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad |2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{2.1}$$

In quantum mechanics these basis vectors may correspond to the energy levels of a three-level atom (or any other three-level system). If $g, h, f \in G$ ($G = SU_3$ or $SL_3(\mathbb{C})$) with $f = g \circ h$ then a representation of the Lie group G in $\mathbb{C}^n$ is a mapping $\Xi\colon G \to \mathrm{Aut}\left(\mathbb{C}^n\right)$ which preserves group multiplication (here $\mathrm{Aut}\left(\mathbb{C}^n\right)$ is the set of all linear invertible maps of the vector space $\mathbb{C}^n$ thus of all regular complex $n \times n$ matrices)

$$\forall g, h \quad f \in G \qquad f = g \circ h \quad \Rightarrow \quad \Xi(f) = \Xi(g)\,\Xi(h). \tag{2.2}$$

As the identity in $\mathrm{Aut}\left(\mathbb{C}^3\right)$ maps every group element of SU_3 and $SL_3(\mathbb{C})$ onto itself one uses the term defining representation. Representations of unitary groups as SU_3 will always be unitary in the following. In this paper our focus lies on properties of the Lie group SU_3 and its Lie algebra $\boldsymbol{su}_3$. It is helpful, however, to embed SU_3 and $\boldsymbol{su}_3$ in some larger groups or algebras to have a deeper insight into their structure.

2.2. *The Lie algebras $\boldsymbol{gl}_3(\mathbb{C})$, $\boldsymbol{sl}_3(\mathbb{C})$, $\boldsymbol{u}_3$ and $\boldsymbol{su}_3$*

In order to understand the structure of the Lie algebras $\boldsymbol{sl}_3(\mathbb{C}) = \boldsymbol{T}_e SL_3(\mathbb{C})$ (where $\boldsymbol{T}_e$ stands for the tangent space at the unit element e of the group) and $\boldsymbol{su}_3 = \boldsymbol{T}_e SU_3$, it is convenient first to consider the Lie algebras $\boldsymbol{gl}_3(\mathbb{C}) = \boldsymbol{T}_e GL_3(\mathbb{C})$ and $\boldsymbol{u}_3 = \boldsymbol{T}_e U_3$ ($GL_3(\mathbb{C})$ is the group of complex regular 3×3 matrices). It is easy to see that the exponential $\exp(A)$ for any complex 3×3 matrix A is a regular matrix $\exp(A) \in GL_3(\mathbb{C})$ with $\exp(A)^{-1} = \exp(-A)$. Thus in the defining representation $\boldsymbol{gl}_3(\mathbb{C})$ is a complex vector space with complex dimension $\dim_{\mathbb{C}} = 9$ spanned by matrices

$$S_{ij} = |i\rangle\langle j| \qquad i, j = 1, 2, 3. \tag{2.3}$$

In the quantum mechanics of a three-level atom these matrices describe a transition from the level $|j\rangle$ to the level $|i\rangle$ and their expectation values have the meaning of a complex polarization for the non-diagonal matrices and of the occupation number for the diagonal matrices. Of course these interpretations go through as well for a set of N three-level atoms where S_{ij} has to be replaced by its (in general reducible) representation $\Xi_N(S_{ij})$ in $\mathbb{C}^{3N}$

$$\Xi_N(S_{ij}) = \sum_{\mu=1}^{N} S_{ij}^{\mu} \tag{2.4}$$

where S_{ij}^{μ} acts as the matrix S_{ij} on the three levels of the μth atom and as an identity on the rest.

It is easily seen that the matrices S_{ij} (and thus also their representatives $\Xi(S_{ij})$) obey the commutation relations

$$[S_{ij}, S_{kl}] = \delta_{kj} S_{il} - \delta_{il} S_{kj} \tag{2.5}$$

and their Hermitian conjugates fulfil

$$S_{ij}^{\dagger} = S_{ji}. \tag{2.6}$$

It is often better to think of S_{ij} as abstract objects which obey (2.5) and (2.6) (we shall also have to consider products $S_{ij} S_{kl}$ from time to time and these have special properties in the defining representation which are not preserved in general representations of $\boldsymbol{gl}_3(\mathbb{C})$).

The Lie algebra $\boldsymbol{u}_3$ is the subalgebra of anti-Hermitian matrices in $\boldsymbol{gl}_3(\mathbb{C})$. Then $\exp(A)$ is unitary for $A \in \boldsymbol{u}_3$ and $\boldsymbol{u}_3$ is a real vector space of real dimension $\dim_{\mathbb{R}}(\boldsymbol{u}_3) = 9$. The algebras $\boldsymbol{gl}_3(\mathbb{C})$ and $\boldsymbol{u}_3$ split into two invariant (normal) subalgebras

$$\begin{aligned} \boldsymbol{gl}_3(\mathbb{C}) &= \boldsymbol{gl}_1(\mathbb{C}) \oplus \boldsymbol{sl}_3(\mathbb{C}) \\ \boldsymbol{u}_3 &= \boldsymbol{u}_1 \oplus \boldsymbol{su}_3 \end{aligned} \tag{2.7}$$

where $\boldsymbol{gl}_1(\mathbb{C}) \equiv \mathbb{C}$ and $\boldsymbol{u}_1 \equiv S^1$ are vector spaces of dimension $\dim_{\mathbb{C}}(\boldsymbol{gl}_1) = 1$ and $\dim_{\mathbb{R}}(\boldsymbol{u}_1) = 1$ spanned by

$$N = S_{11} + S_{22} + S_{33}. \tag{2.8}$$

In a quantum-mechanical model for many three-level atoms N is the operator that counts the number of atoms. The Lie algebra $\boldsymbol{sl}_3(\mathbb{C})$ is the subalgebra of $\boldsymbol{gl}_3(\mathbb{C})$ such that $\operatorname{tr} A = 0$ for $A \in \boldsymbol{sl}_3(\mathbb{C})$, i.e. $\det\exp(A) = 1$. It is spanned by the six generators S_{ij} with $i \neq j$ and

$$H_1 = \tfrac{1}{2}(S_{11} - S_{22}) \qquad H_2 = \tfrac{1}{2}(S_{22} - S_{33}) \tag{2.9}$$

and has complex dimension $\dim_{\mathbb{C}}(\boldsymbol{sl}_3(\mathbb{C})) = 8$. The two operators H_1 and H_2 count the population differences between the pairs of levels $(|1\rangle, |2\rangle)$ and $(|2\rangle, |3\rangle)$. Finally, $\boldsymbol{su}_3 \subset \boldsymbol{sl}_3(\mathbb{C})$ is the real Lie subalgebra of anti-Hermitian generators and has real dimension $\dim_{\mathbb{R}}(\boldsymbol{su}_3) = 8$. The generators H_1 and H_2 span the so-called Cartan subalgebra (maximal commutative subalgebra [18]) with

$$[H_1, H_2] = 0. \tag{2.10}$$

We shall call the generators S_{ij} raising operators if $i < j$ and lowering operators if $i > j$.

2.3. Some helpful disentangling formulae

In the definition of SU_3 coherent states the following group element $b_-[\gamma] \in SL_3(\mathbb{C})$ depending on three complex numbers $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$ will be frequently used

$$b_-[\gamma] := \exp\left(\gamma_1 S_{21} + \gamma_2 S_{32} + \left(\gamma_3 - \tfrac{1}{2}\gamma_1\gamma_2\right)S_{31}\right) = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_3 & \gamma_2 & 1 \end{pmatrix} \tag{2.11}$$

with the shorthand $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{C}^3$.

In fact, the above group element belongs to the subgroup of lower triangular matrices in $SL_3(\mathbb{C})$ having unit diagonal elements. This subgroup is a (non-unitary) representation of the Heisenberg–Weyl group [18] and is generated by the lowering operators S_{ij} $(i > j)$. It will be denoted B_-. The following disentangling formula for $b_-[\gamma]$ is derived easily by an explicit calculations in the defining representation:

$$\begin{aligned} b_-[\gamma] &= \exp((\gamma_3 - \gamma_1\gamma_2)S_{31}) \exp(\gamma_2 S_{32}) \exp(\gamma_1 S_{21}) \\ &= \exp(\gamma_3 S_{31}) \exp(\gamma_1 S_{21}) \exp(\gamma_2 S_{32}). \end{aligned} \tag{2.12}$$

As $b_-[\gamma]$ is a lower triangular matrix its Hermitian conjugate $b_-[\gamma]^\dagger$ is an upper triangular matrix $(b_-[\gamma]^\dagger \in B_+)$. In the product $b_-[\gamma]^\dagger \circ b_-[\gamma]$ all lowering operators are to the right of all raising operators. It is sometimes useful to reorder this product in such a way that all the raising operators are on the right and all the lowering operators on the left

$$\begin{aligned} b_-[\gamma]^\dagger \circ b_-[\gamma] &= b_-[\alpha] \circ h_{diag} \circ b_-[\alpha]^\dagger \\ &= \exp\left(\alpha_1 S_{21} + \alpha_2 S_{32} + \left(\alpha_3 - \tfrac{1}{2}\alpha_1\alpha_2\right)S_{31}\right) \circ \exp(2F_1 H_1 + 2F_2 H_2) \\ &\quad \circ \exp\left(\alpha_1^* S_{12} + \alpha_2^* S_{23} + \left(\alpha_3^* - \tfrac{1}{2}\alpha_1^*\alpha_2^*\right)S_{13}\right) \end{aligned} \tag{2.13}$$

$$
\begin{aligned}
F_1 &:= \log f_1 \\
F_2 &:= \log f_2 \\
f_1 &:= 1 + |\gamma_1|^2 + |\gamma_3|^2 \\
f_2 &:= 1 + |\gamma_2|^2 + |\gamma_3 - \gamma_1\gamma_2|^2 \\
\alpha_1 &:= \frac{1}{f_1}(\gamma_1 + \gamma_2^*\gamma_3) \\
\alpha_2 &:= \frac{1}{f_2}(\gamma_2 - \gamma_1^*(\gamma_3 - \gamma_1\gamma_2)) \\
\alpha_3 &:= \frac{1}{f_1}\gamma_3
\end{aligned}
\tag{2.14}
$$

which again can be proved by an explicit calculation in the defining representation (see also [19–21]).

2.4. Parametrization of $SL_3(\mathbb{C})$ and SU_3

A parametrization of the Lie group SU_3 suitable for the topics to be discussed in this paper is obtained via so-called Gauss decomposition [18]. Almost every (with respect to the Haar measure) 3×3 matrix $A \in SL_3(\mathbb{C})$ can be expressed as a product of three matrices

$$A = b_- \circ d \circ b_+ \tag{2.15}$$

where $b_- \in B_-$, $b_+ \in B_+$ and d is a diagonal matrix with unit determinant. A useful parametrization of the subgroup $SU_3 \subset SL_3(\mathbb{C})$ valid for almost all matrices in SU_3 is thus achieved by requiring A to be unitary. Shifting the phases resulting from the matrix d to the right we obtain the following parametrization of a group element $g[\gamma; \psi_1, \psi_2] \in SU_3$:

$$
\begin{aligned}
g[\gamma; \psi_1, \psi_2] = {} & \exp\left(\gamma_1 S_{21} + \gamma_2 S_{32} + \left(\gamma_3 - \tfrac{1}{2}\gamma_1\gamma_2\right)S_{31}\right) \circ \exp(-F_1 H_1 - F_2 H_2) \\
& \circ \exp\left(\beta_1^* S_{12} + \beta_2^* S_{23} + \left(\beta_3^* - \tfrac{1}{2}\beta_1^*\beta_2^*\right)S_{13}\right) \circ \exp(\mathrm{i}\psi_1 H_1 + \mathrm{i}\psi_2 H_2).
\end{aligned}
\tag{2.16}
$$

If we include all kinds of limits $|\gamma_i| \to \infty$ every group element is reached. Here F_1 and F_2 are real functions of γ given by (2.14). The parameter γ ranges in $\mathbb{C}^3$ and the real angles $\psi_{1,2}$ in $[0, 4\pi)$ as $H_{1,2}$ has half-integer eigenvalues. As a consequence of the unitarity of A the coefficients β are complex functions of γ

$$
\begin{aligned}
\beta_1 &= -(\gamma_1 + \gamma_2^*\gamma_3)\frac{1}{\sqrt{f_2}} \\
\beta_2 &= \left(\gamma_1^*\gamma_3 - \gamma_2\left(1 + |\gamma_1|^2\right)\right)\frac{1}{\sqrt{f_1}} \\
\beta_3 &= -(\gamma_3 - \gamma_1\gamma_2)\sqrt{\frac{f_1}{f_2}}.
\end{aligned}
\tag{2.17}
$$

The above relations can be easily inverted giving the parameters γ_i as functions of β_j,

$$
\begin{aligned}
\gamma_1 &= -(\beta_1 + \beta_2^*\beta_3)\frac{1}{\sqrt{f_2}} \\
\gamma_2 &= \left(\beta_1^*\beta_3 - \beta_2\left(1 + |\beta_1|^2\right)\right)\frac{1}{\sqrt{f_1}} \\
\gamma_3 &= -(\beta_3 - \beta_1\beta_2)\sqrt{\frac{f_1}{f_2}}.
\end{aligned}
\tag{2.18}
$$

Note the symmetry between (2.17) and (2.18). The matrix $g[\gamma; \psi_1, \psi_2]$ in the defining representation is given explicitly by

$$
\begin{aligned}
g[\gamma; \psi_1, \psi_2] &= \begin{pmatrix} 1 & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_3 & \gamma_2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1/\sqrt{f1} & 0 & 0 \\ 0 & \sqrt{f_1/f_2} & 0 \\ 0 & 0 & \sqrt{f_2} \end{pmatrix} \circ \begin{pmatrix} 1 & \beta_1^* & \beta_3^* \\ 0 & 1 & \beta_2^* \\ 0 & 0 & 1 \end{pmatrix} \\
&\quad \circ \begin{pmatrix} \exp(\mathrm{i}\psi_1/2) & 0 & 0 \\ 0 & \exp(\mathrm{i}(\psi_2-\psi_1)/2) & 0 \\ 0 & 0 & \exp(-\mathrm{i}\psi_2/2) \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\mathrm{e}^{\mathrm{i}\psi_1/2}}{\sqrt{f_1}} & -\dfrac{(\gamma_1^* + \gamma_2\gamma_3^*)\mathrm{e}^{\mathrm{i}(\psi_2-\psi_1)/2}}{\sqrt{f_1 f_2}} & -\dfrac{(\gamma_3^* - \gamma_1^*\gamma_2^*)\mathrm{e}^{-\mathrm{i}\psi_2/2}}{\sqrt{f_2}} \\ \dfrac{\gamma_1\mathrm{e}^{\mathrm{i}\psi_1/2}}{\sqrt{f_1}} & \dfrac{(1+|\gamma_3|^2 - \gamma_1\gamma_2\gamma_3^*)\mathrm{e}^{\mathrm{i}(\psi_2-\psi_1)/2}}{\sqrt{f_1 f_2}} & -\dfrac{\gamma_2^*\mathrm{e}^{-\mathrm{i}\psi_2/2}}{\sqrt{f_2}} \\ \dfrac{\gamma_3\mathrm{e}^{\mathrm{i}\psi_1/2}}{\sqrt{f_1}} & \dfrac{(\gamma_2 - \gamma_1^*\gamma_3 + \gamma_2|\gamma_1|^2)\mathrm{e}^{\mathrm{i}(\psi_2-\psi_1)/2}}{\sqrt{f_1 f_2}} & \dfrac{\mathrm{e}^{-\mathrm{i}\psi_2/2}}{\sqrt{f_2}} \end{pmatrix}.
\end{aligned}
\tag{2.19}
$$

2.5. SU_3 group multiplication

By direct calculations it is possible to derive the group multiplication formulae of SU_3 in the parametrization given in the last section

$$
g[\gamma''; \psi_1'', \psi_2''] = g[\gamma'; \psi_1', \psi_2'] \circ g[\gamma; \psi_1, \psi_2]. \tag{2.20}
$$

Comparing the matrix on the left-hand side with the one on the right-hand side yields

$$
\begin{aligned}
\psi_1'' &= \psi_1 + \psi_1' - 2\mathrm{i}\log \\
&\qquad \times \frac{\sqrt{f_2'} - (\gamma_1'^* + \gamma_2'\gamma_3'^*)\gamma_1\mathrm{e}^{-\mathrm{i}(2\psi_1'-\psi_2')/2} - (\gamma_3'^* - \gamma_1'^*\gamma_2'^*)\gamma_3\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\sqrt{f_1'}}{\left|\sqrt{f_2'} - (\gamma_1'^* + \gamma_2'\gamma_3'^*)\gamma_1\mathrm{e}^{-\mathrm{i}(2\psi_1'-\psi_2')/2} - (\gamma_3'^* - \gamma_1'^*\gamma_2'^*)\gamma_3\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\sqrt{f_1'}\right|} \\
&= \psi_1 + \psi_1' - 2\mathrm{i}\log \frac{1 + \beta_1'^*\gamma_1\mathrm{e}^{\mathrm{i}(\psi_2'-2\psi_1')/2} + \beta_3'^*\gamma_3\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}}{\left|1 + \beta_1'^*\gamma_1\mathrm{e}^{\mathrm{i}(\psi_2'-2\psi_1')/2} + \beta_3'^*\gamma_3\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\right|} \\
\psi_2'' &= \psi_2 + \psi_2' - 2\mathrm{i}\log \\
&\quad \times \frac{\sqrt{f_1'} - \left(\gamma_2'^* - \gamma_1'\gamma_3'^* + \gamma_2'^*|\gamma_1'|^2\right)\gamma_2\mathrm{e}^{-\mathrm{i}(2\psi_2'-\psi_1')/2} - \gamma_3'^*(\gamma_3 - \gamma_1\gamma_2)\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\sqrt{f_2'}}{\left|\sqrt{f_1'} - \left(\gamma_2'^* - \gamma_1'\gamma_3'^* + \gamma_2'^*|\gamma_1'|^2\right)\gamma_2\mathrm{e}^{-\mathrm{i}(2\psi_2'-\psi_1')/2} - \gamma_3'^*(\gamma_3 - \gamma_1\gamma_2)\mathrm{e}^{-\mathrm{i}\psi_1'+\psi_2'/2}\sqrt{f_2'}\right|} \\
&= \psi_2 + \psi_2' - 2\mathrm{i}\log \frac{1 + \beta_2'^*\gamma_2\mathrm{e}^{-\mathrm{i}(2\psi_2'-\psi_1')/2} + (\beta_3'^* - \beta_1'^*\beta_2'^*)(\gamma_3 - \gamma_1\gamma_2)\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}}{\left|1 + \beta_2'^*\gamma_2\mathrm{e}^{-\mathrm{i}(2\psi_2'-\psi_1')/2} + (\beta_3'^* - \beta_1'^*\beta_2'^*)(\gamma_3 - \gamma_1\gamma_2)\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\right|} \\
\gamma_1'' &= \frac{\sqrt{f_2'}\gamma_1' + \left(1 + |\gamma_3'|^2 - \gamma_1'\gamma_2'\gamma_3'^*\right)\gamma_1\mathrm{e}^{\mathrm{i}(\psi_2'-2\psi_1')/2} - \sqrt{f_1'}\gamma_2'^*\gamma_3\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}}{\sqrt{f_2'} - (\gamma_1'^* + \gamma_2'\gamma_3'^*)\gamma_1\mathrm{e}^{\mathrm{i}(\psi_2'-2\psi_1')/2} - \sqrt{f_1'}(\gamma_3'^* - \gamma_1'^*\gamma_2'^*)\gamma_3\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}} \\
&= \Big[-(\beta_1' + \beta_2'^*\beta_3') + \left(1 + |\beta_3'|^2 - \beta_1'^*\beta_2'^*\beta_3'\right)\gamma_1\mathrm{e}^{\mathrm{i}(\psi_2'-2\psi_1')/2} \\
&\qquad + \left(\beta_2'^* + \beta_2'^*|\beta_1'|^2 - \beta_1'\beta_3'^*\right)\gamma_3\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\Big] \\
&\qquad \times\left[\sqrt{f_2'}\left(1 + \beta_1'^*\gamma_1\mathrm{e}^{\mathrm{i}(\psi_2'-2\psi_1')/2} + \beta_3'^*\gamma_3\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\right)\right]^{-1}
\end{aligned}
\tag{2.21}
$$

$$\begin{aligned}
\gamma_2'' &= \frac{\sqrt{f_1'}\gamma_2' + \left(1+|\gamma_3'|^2 - \gamma_1'^*\gamma_2'^*\gamma_3'\right)\gamma_2 \mathrm{e}^{-\mathrm{i}(2\psi_2'-\psi_1')/2} + \sqrt{f_2'}\gamma_1'^*(\gamma_3-\gamma_1\gamma_2)\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}}{\sqrt{f_1'} + \left(\gamma_1'\gamma_3'^* - \gamma_2'^* - \gamma_2'^*|\gamma_1'|^2\right)\gamma_2 \mathrm{e}^{-\mathrm{i}(2\psi_2'-\psi_1')/2} - \sqrt{f_2'}\gamma_3'^*(\gamma_3-\gamma_1\gamma_2)\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}} \\
&= \Big[\left(\beta_1'^*\beta_3' - \beta_2' - \beta_2'|\beta_1'|^2\right) + \left(1+|\beta_3'|^2 - \beta_1'\beta_2'\beta_3'^*\right)\gamma_2 \mathrm{e}^{-\mathrm{i}(2\psi_2'-\psi_1')/2} \\
&\qquad -(\beta_1'^* + \beta_2'\beta_3'^*)(\gamma_3-\gamma_1\gamma_2)\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\Big] \\
&\qquad \times\left[\sqrt{f_1'}\left(1+\beta_2'^*\gamma_2 \mathrm{e}^{-\mathrm{i}(2\psi_2'-\psi_1')/2} + (\beta_3'^* - \beta_1'^*\beta_2'^*)(\gamma_3-\gamma_1\gamma_2)\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\right)\right]^{-1} \\
\gamma_3'' &= \frac{\sqrt{f_2'}\gamma_3' - \left(\gamma_1'^*\gamma_3' - \gamma_2' - \gamma_2'|\gamma_1'|^2\right)\gamma_1 \mathrm{e}^{\mathrm{i}(\psi_2'-2\psi_1')/2} + \sqrt{f_1'}\gamma_3 \mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}}{\sqrt{f_2'} - (\gamma_1'^* + \gamma_2'\gamma_3'^*)\gamma_1 \mathrm{e}^{\mathrm{i}(\psi_2'-2\psi_1')/2} - \sqrt{f_1'}(\gamma_3'^* - \gamma_1'^*\gamma_2'^*)\gamma_3 \mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}} \\
&= \sqrt{\frac{f_1'}{f_2'}}\frac{-(\beta_3' - \beta_1'\beta_2') - \beta_2'\gamma_1 \mathrm{e}^{\mathrm{i}(\psi_2'-2\psi_1')/2} + \gamma_3 \mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}}{1+\beta_1'^*\gamma_1 \mathrm{e}^{\mathrm{i}(\psi_2'-2\psi_1')/2} + \beta_3'^*\gamma_3 \mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}} \\
f_1'' &= \frac{f_1' f_2' f_1}{\left|\sqrt{f_2'} - (\gamma_1'^* + \gamma_2'\gamma_3'^*)\gamma_1 \mathrm{e}^{-\mathrm{i}(2\psi_1'-\psi_2')/2} - (\gamma_3'^* - \gamma_1'^*\gamma_2'^*)\gamma_3 \mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\sqrt{f_1'}\right|^2} \\
&= \frac{f_1' f_1}{\left|1+\beta_1'^*\gamma_1 \mathrm{e}^{\mathrm{i}(\psi_2'-2\psi_1')/2} + \beta_3'^*\gamma_3 \mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\right|^2} \\
f_2'' &= \frac{f_1' f_2' f_2}{\left|\sqrt{f_1'} - \left(\gamma_2'^* - \gamma_1'\gamma_3'^* + \gamma_2'^*|\gamma_1'|^2\right)\gamma_2 \mathrm{e}^{-\mathrm{i}(2\psi_2'-\psi_1')/2} - \gamma_3'^*(\gamma_3-\gamma_1\gamma_2)\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\sqrt{f_2'}\right|^2} \\
&= \frac{f_2' f_2}{\left|1+\beta_2'^*\gamma_2 \mathrm{e}^{-\mathrm{i}(2\psi_2'-\psi_1')/2} + (\beta_3'^* - \beta_1'^*\beta_2'^*)(\gamma_3-\gamma_1\gamma_2)\mathrm{e}^{-\mathrm{i}(\psi_1'+\psi_2')/2}\right|^2}.
\end{aligned}$$

We shall need the above group multiplication formulae to calculate the projection of one coherent state upon another and the action of a group element of SU_3 on a coherent state. The inverse of a group element of the form $g[\gamma] := g[\gamma;\, \psi_1 = 0,\, \psi_2 = 0]$ is easily seen to be

$$g[\gamma]^{-1} = g[\gamma]^\dagger = g[\beta]. \tag{2.22}$$

2.6. Invariant 1-forms, invariant vector fields and the Haar measure on SU_3

There is a basis of eight independent invariant 1-forms on SU_3. Any invariant differential form on SU_3 may be expressed as linear combinations of outer products of these. In particular, the Haar measure (the invariant volume) on SU_3 is a differential 8-form. Up to a constant factor the volume form is given by the outer product of all basis 1-forms. As SU_3 is a matrix group such a basis of invariant 1-forms is given by the independent matrix elements of the matrix Ω defined as [22]

$$\Omega = g[\gamma;\, \psi_1, \psi_2]^{-1}\, \mathrm{d}g[\gamma;\, \psi_1, \psi_2]. \tag{2.23}$$

This is an anti-Hermitian and traceless matrix of 1-forms. Explicitly its elements are given by

$$\begin{aligned}
\Omega_{11} &= \frac{\mathrm{i}}{2}\alpha_{\psi_1} \\
&= \frac{1}{2}\mathrm{i}\,\mathrm{d}\psi_1 + \frac{\gamma_1^*}{2f_1}\mathrm{d}\gamma_1 - \frac{\gamma_1}{2f_1}\mathrm{d}\gamma_1^* + \frac{\gamma_3^*}{2f_1}\mathrm{d}\gamma_3 - \frac{\gamma_3}{2f_1}\mathrm{d}\gamma_3^* \\
\Omega_{22} &= -\Omega_{11} - \Omega_{33}
\end{aligned}$$

$$
\begin{aligned}
\Omega_{33} &= -\frac{\mathrm{i}}{2}\alpha_{\psi_2} \\
&= -\frac{1}{2}\mathrm{i}\,\mathrm{d}\psi_2 + \frac{\gamma_2(\gamma_3^* - \gamma_1^*\gamma_2^*)}{2f_2}\mathrm{d}\gamma_1 + \frac{\gamma_1\gamma_3^* - \gamma_2^*\left(1+|\gamma_1|^2\right)}{2f_2}\mathrm{d}\gamma_2 - \frac{\gamma_3^* - \gamma_1^*\gamma_2^*}{2f_2}\mathrm{d}\gamma_3 \\
&\qquad - \frac{\gamma_2^*(\gamma_3 - \gamma_1\gamma_2)}{2f_2}\mathrm{d}\gamma_1^* - \frac{\gamma_1^*\gamma_3 - \gamma_2\left(1+|\gamma_1|^2\right)}{2f_2}\mathrm{d}\gamma_2^* + \frac{\gamma_3 - \gamma_1\gamma_2}{2f_2}\mathrm{d}\gamma_3^* \qquad (2.24) \\
\Omega_{12} &= -\alpha_{\gamma_1^*} \\
&= \frac{1}{f_1\sqrt{f_2}}\mathrm{e}^{\mathrm{i}(\psi_2 - 2\psi_1)/2}\left(-\left(1+|\gamma_3|^2 - \gamma_1\gamma_2\gamma_3^*\right)\mathrm{d}\gamma_1^* + \left(\gamma_1^*\gamma_3 - \gamma_2 - |\gamma_1|^2\gamma_2\right)\mathrm{d}\gamma_3^*\right) \\
\Omega_{13} &= -\alpha_{\gamma_3^*} \\
&= \frac{1}{\sqrt{f_1 f_2}}\mathrm{e}^{-\mathrm{i}(\psi_1+\psi_2)/2}(\gamma_2^*\,\mathrm{d}\gamma_1^* - \mathrm{d}\gamma_3^*) \\
\Omega_{23} &= -\alpha_{\gamma_2^*} \\
&= \frac{1}{f_2\sqrt{f_1}}\mathrm{e}^{-\mathrm{i}(2\psi_2-\psi_1)/2}(-\gamma_2^*(\gamma_1 + \gamma_2^*\gamma_3)\,\mathrm{d}\gamma_1^* - f_1\,\mathrm{d}\gamma_2^* + (\gamma_1 + \gamma_2^*\gamma_3)\,\mathrm{d}\gamma_3^*) \\
\Omega_{21} &= \alpha_{\gamma_1} = -\Omega_{12}^* \\
\Omega_{31} &= \alpha_{\gamma_3} = -\Omega_{13}^* \\
\Omega_{32} &= \alpha_{\gamma_2} = -\Omega_{23}^*
\end{aligned}
$$

where we have introduced the 1-forms α_x which reduce to $\alpha_x|_e = \mathrm{d}x$ at the identity ($\gamma = 0$, $\psi = 0$) for x being one of the eight parameters γ_i, ψ_i. Thus the Haar measure μ on SU_3 reads

$$
\begin{aligned}
\mu &= -\mathrm{i}c\Omega_{33}\wedge\Omega_{11}\wedge\Omega_{21}\wedge\Omega_{12}\wedge\Omega_{31}\wedge\Omega_{13}\wedge\Omega_{32}\wedge\Omega_{23} \\
&= -\mathrm{i}c\frac{1}{4f_1^2f_2^2}\,\mathrm{d}\psi_1\wedge\mathrm{d}\psi_2\wedge\mathrm{d}\gamma_1\wedge\mathrm{d}\gamma_1^*\wedge\mathrm{d}\gamma_2\wedge\mathrm{d}\gamma_2^*\wedge\mathrm{d}\gamma_3\wedge\mathrm{d}\gamma_3^* \qquad (2.25)
\end{aligned}
$$

where the real constant c is chosen in such a way that the volume of the group is normalized to unity

$$\int \mu = 1. \qquad (2.26)$$

Noting that the angles ψ_1 and ψ_2 range in $[0, 4\pi)$ and using

$$\int \mathrm{d}^2\gamma_1 \int \mathrm{d}^2\gamma_2 \int \mathrm{d}^2\gamma_3 \frac{1}{f_1^2f_2^2} = \frac{\pi^3}{2} \qquad (2.27)$$

(with $\mathrm{d}^2\gamma_j = \mathrm{d}(\mathrm{Re}\,\gamma_j)\mathrm{d}(\mathrm{Im}\,\gamma_j) = \frac{1}{2\mathrm{i}}\mathrm{d}\gamma_j\wedge\mathrm{d}\gamma_j^*$) one obtains

$$c = \frac{1}{16\pi^5}. \qquad (2.28)$$

There are also eight invariant vector fields V_x on SU_3 which may be chosen such that

$$\alpha_x(V_y) = \delta_{xy}. \qquad (2.29)$$

These are given by

$$
\begin{aligned}
V_{\psi_1} &= \frac{\partial}{\partial\psi_1} \\
V_{\psi_2} &= \frac{\partial}{\partial\psi_2} \\
V_{\gamma_1} &= \mathrm{i}\frac{\gamma_1^* + \gamma_2\gamma_3^*}{\sqrt{f_2}}\mathrm{e}^{-\mathrm{i}(2\psi_1-\psi_2)/2}\frac{\partial}{\partial\psi_1} + \frac{f_1}{\sqrt{f_2}}\mathrm{e}^{-\mathrm{i}(2\psi_1-\psi_2)/2}\frac{\partial}{\partial\gamma_1} + \frac{\gamma_2 f_1}{\sqrt{f_2}}\mathrm{e}^{-\mathrm{i}(2\psi_1-\psi_2)/2}\frac{\partial}{\partial\gamma_3} \\
V_{\gamma_2} &= \mathrm{i}\frac{\left(\gamma_2^* + \gamma_2^*|\gamma_1|^2 - \gamma_1\gamma_3^*\right)}{\sqrt{f_1}}\mathrm{e}^{-\mathrm{i}(2\psi_2-\psi_1)/2}\frac{\partial}{\partial\psi_2} + \frac{f_2}{\sqrt{f_1}}\mathrm{e}^{-\mathrm{i}(2\psi_2-\psi_1)/2}\frac{\partial}{\partial\gamma_2} \\
V_{\gamma_3} &= \mathrm{i}\frac{(\gamma_3^* - \gamma_1^*\gamma_2^*)\sqrt{f_1}}{\sqrt{f_2}}\mathrm{e}^{-\mathrm{i}(\psi_1+\psi_2)/2}\frac{\partial}{\partial\psi_1} + \mathrm{i}\frac{\gamma_3^*\sqrt{f_2}}{\sqrt{f_1}}\mathrm{e}^{-\mathrm{i}(\psi_1+\psi_2)/2}\frac{\partial}{\partial\psi_2} \\
&\quad - \frac{\left(\gamma_2^* + \gamma_2^*|\gamma_1|^2 - \gamma_1\gamma_3^*\right)\sqrt{f_1}}{\sqrt{f_2}}\mathrm{e}^{-\mathrm{i}(\psi_1+\psi_2)/2}\frac{\partial}{\partial\gamma_1} \\
&\quad + \frac{(\gamma_1^* + \gamma_2\gamma_3^*)\sqrt{f_2}}{\sqrt{f_1}}\mathrm{e}^{-\mathrm{i}(\psi_1+\psi_2)/2}\frac{\partial}{\partial\gamma_2} \\
&\quad + \frac{\left(1 + |\gamma_3|^2 - \gamma_1^*\gamma_2^*\gamma_3\right)\sqrt{f_1}}{\sqrt{f_2}}\mathrm{e}^{-\mathrm{i}(\psi_1+\psi_2)/2}\frac{\partial}{\partial\gamma_3} \\
V_{\gamma_{1^*}} &= V_{\gamma_1}^* \\
V_{\gamma_{2^*}} &= V_{\gamma_2}^* \\
V_{\gamma_{3^*}} &= V_{\gamma_3}^*.
\end{aligned}
\tag{2.30}
$$

2.7. Irreducible representations of SU_3

We shall only sketch the most important facts about unitary irreducible representations of SU_3 [18]. An irreducible representation of a group G on a Hilbert space $\mathcal{H}$

$$\Xi:\quad G \to \mathrm{Aut}(\mathcal{H}) \tag{2.31}$$

induces, in a natural way, a representation of the Lie algebra $\boldsymbol{g} = \boldsymbol{T}_e G$

$$\xi := \boldsymbol{T}_e\Xi:\quad \boldsymbol{g} \to \mathrm{End}(\mathcal{H}) \tag{2.32}$$

where $\boldsymbol{T}_e\Xi$ denotes the derivative of Ξ at the identity and $\mathrm{End}(\mathcal{H})$ stands for the space of all linear mappings of $\mathcal{H}$ into itself. When a group element $g \in G$ acts on a vector $|\nu\rangle \in \mathcal{H}$ we shall write, for brevity, $g|\nu\rangle$ instead of $\Xi(g)|\nu\rangle$ (and use an analogous convention for elements of the algebra). As the group $G = SU_3$ is compact its all irreducible representations are finite dimensional [18]. In the Hilbert space $\mathcal{H}$ of any such representation the vector of highest weight $|\mu\rangle$ is characterized by [18]

$$S_{23}|\mu\rangle = S_{13}|\mu\rangle = S_{12}|\mu\rangle = 0. \tag{2.33}$$

We shall assume $|\mu\rangle$ to be normalized. The vector of highest weight is unique up to a phase factor and it is a common eigenvector of the Cartan subalgebra generators of $\boldsymbol{su}_3$

$$H_1|\mu\rangle = \tfrac{1}{2}\lambda_1|\mu\rangle \qquad H_2|\mu\rangle = \tfrac{1}{2}\lambda_2|\mu\rangle. \tag{2.34}$$

The two numbers λ_i are positive integers which characterize an irreducible representation uniquely. We shall call it the $[\lambda_1, \lambda_2]$-representation. Its dimension is [18]

$$\dim\mathcal{H}_{[\lambda_1,\lambda_2]} = \tfrac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2). \tag{2.35}$$

Each representation induces, also in a natural way, a representation of $SL_3(\mathbb{C})$, of its Lie algebra $\boldsymbol{sl}_3(\mathbb{C})$, and of the universal enveloping algebra of $\boldsymbol{sl}_3(\mathbb{C})$ (the algebra of formal products of elements of $\boldsymbol{sl}_3(\mathbb{C})$).

3. SU_3 coherent states

3.1. Definition of SU_3 coherent states

The coherent states are a special overcomplete basis in the Hilbert space of the $[\lambda_1, \lambda_2]$-representation of SU_3. Their properties are neatly connected to the Lie groups SU_3 and $SL_3(\mathbb{C})$. If $|\mu\rangle$ is the highest weight vector (2.33) and if both λ_1 and λ_2 are non-vanishing then a coherent state $\|\gamma\rangle$ is defined as [14]

$$\|\gamma\rangle := b_-[\gamma]|\mu_{[\lambda_1,\lambda_2]}\rangle \tag{3.1}$$

where $b_-[\gamma]$ is given by (2.12). Such states are not normalized. Using (2.13), (2.33), and (2.34) the squared norm of $\|\gamma\rangle$ is calculated to be

$$\langle\gamma\|\gamma\rangle = f_1^{\lambda_1} f_2^{\lambda_2}. \tag{3.2}$$

The logarithm of this norm will play the role of a generating function for the symplectic structure calculated in section 4

$$\mathcal{F} = \log\langle\gamma\|\gamma\rangle = \lambda_1 \log(f_1) + \lambda_2 \log(f_2). \tag{3.3}$$

The normalized coherent states are given by

$$|\gamma\rangle = f_1^{-\lambda_1/2} f_2^{-\lambda_2/2}\|\gamma\rangle = g[\gamma]|\mu\rangle \tag{3.4}$$

hence $|\mu\rangle = |\gamma = 0\rangle$. We will allow the limit $|\gamma_i| \to \infty$ for coherent states.

If either $\lambda_1 = 0$ or $\lambda_2 = 0$ the definition (3.1) would still give well defined coherent states [23]. However, there would be some degeneracy which we would like to avoid. One may easily calculate that all elements of the Heisenberg–Weyl group B_- of the form $b_-[\gamma_1 = 0, \gamma_2 = z, \gamma_3 = 0]$ leave the highest weight vector in a $[\lambda_1, \lambda_2 = 0]$-representation invariant

$$b_-[\gamma_1 = 0, \gamma_2 = z, \gamma_3 = 0]|\mu_{[\lambda_1,\lambda_2=0]}\rangle = |\mu_{[\lambda_1,\lambda_2=0]}\rangle. \tag{3.5}$$

In the other case we have

$$b_-[\gamma_1 = z, \gamma_2 = 0, \gamma_3 = 0]|\mu_{[\lambda_1=0,\lambda_2]}\rangle = |\mu_{[\lambda_1=0,\lambda_2]}\rangle. \tag{3.6}$$

We shall call these irreducible representations degenerate and in order to divide out this isotropy we restrict $b_-[\gamma]$ (3.1) to $\gamma_2 = 0$ if $\lambda_1 = 0$ and to $\gamma_1 = 0$ if $\lambda_2 = 0$ (if both λ_i vanish we have the trivial representation which is of no interest here). The definition of normalized coherent states goes through just as before (setting the appropriate variable to zero in (3.2)–(3.4)). All of the following calculations are true in the degenerate representations with the given additional settings if no restriction is explicitly mentioned. We shall now give some useful properties of SU_3 coherent states.

3.2. Action of SU_3 and $\boldsymbol{sl}_3(\mathbb{C})$ on a coherent state

The group multiplication formula in section 2.5 enables us to give the action of a SU_3 Lie group element $g(\gamma'; \psi_1', \psi_2')$ on a coherent state $|\gamma\rangle$ in the $[\lambda_1, \lambda_2]$ representation

$$g(\gamma'; \psi_1', \psi_2')|\gamma\rangle = \exp(\mathrm{i}\psi_1''\lambda_1/2)\exp(\mathrm{i}\psi_2''\lambda_2/2)|\gamma''\rangle. \tag{3.7}$$

The double-primed variables are connected to the single-primed and non-primed ones by the group multiplication formulae (2.20) and (2.21) of the form $g[\gamma''; \psi_1'', \psi_2''] = g[\gamma'; \psi_1', \psi_2'] \circ g[\gamma; \psi_1 = 0, \psi_2 = 0]$. Since the algebra $\boldsymbol{sl}_3(\mathbb{C})$ is in a natural way represented in the same space $\mathcal{H}$, we can consider its action on the coherent states. The action of an element of $\boldsymbol{sl}_3(\mathbb{C})$ on a coherent state does not, in general, yield another coherent state. It is best expressed in terms of differential operators

$$S_{ij}|\gamma\rangle = \Delta_{ij}\left(\gamma, \frac{\partial}{\partial\gamma}\right)|\gamma\rangle. \tag{3.8}$$

Note that in products the order has to be reversed

$$S_{ij}S_{kl}|\gamma\rangle = \Delta_{kl}\left(\gamma, \frac{\partial}{\partial\gamma}\right)\Delta_{ij}\left(\gamma, \frac{\partial}{\partial\gamma}\right)|\gamma\rangle. \tag{3.9}$$

Using the definition of coherent states these differential operators may be calculated explicitly. We give the differential operators for the action on non-normalized coherent states (the normalization is easily introduced)

$$\begin{aligned}
S_{32}\|\gamma\rangle &= \left(\frac{\partial}{\partial\gamma_2} + \gamma_1\frac{\partial}{\partial\gamma_3}\right)\|\gamma\rangle \\
S_{31}\|\gamma\rangle &= \frac{\partial}{\partial\gamma_3}\|\gamma\rangle \\
S_{21}\|\gamma\rangle &= \frac{\partial}{\partial\gamma_1}\|\gamma\rangle \\
S_{23}\|\gamma\rangle &= \left(\gamma_2\lambda_2 + \gamma_3\frac{\partial}{\partial\gamma_1} - \gamma_2^2\frac{\partial}{\partial\gamma_2}\right)\|\gamma\rangle \\
S_{13}\|\gamma\rangle &= \left(\gamma_3\lambda_1 + (\gamma_3 - \gamma_1\gamma_2)\lambda_2 - \gamma_1\gamma_3\frac{\partial}{\partial\gamma_1} - \gamma_2(\gamma_3 - \gamma_1\gamma_2)\frac{\partial}{\partial\gamma_2} - \gamma_3^2\frac{\partial}{\partial\gamma_3}\right)\|\gamma\rangle \\
S_{12}\|\gamma\rangle &= \left(\gamma_1\lambda_1 - \gamma_1^2\frac{\partial}{\partial\gamma_1} - (\gamma_3 - \gamma_1\gamma_2)\frac{\partial}{\partial\gamma_2} - \gamma_1\gamma_3\frac{\partial}{\partial\gamma_3}\right)\|\gamma\rangle \\
H_1\|\gamma\rangle &= \left(\frac{\lambda_1}{2} - \gamma_1\frac{\partial}{\partial\gamma_1} + \frac{1}{2}\gamma_2\frac{\partial}{\partial\gamma_2} - \frac{1}{2}\gamma_3\frac{\partial}{\partial\gamma_3}\right)\|\gamma\rangle \\
H_2\|\gamma\rangle &= \left(\frac{\lambda_2}{2} + \frac{1}{2}\gamma_1\frac{\partial}{\partial\gamma_1} - \gamma_2\frac{\partial}{\partial\gamma_2} - \frac{1}{2}\gamma_3\frac{\partial}{\partial\gamma_3}\right)\|\gamma\rangle.
\end{aligned} \tag{3.10}$$

In the degenerate representations the coherent states depend only on two of the variables γ_i. The partial derivatives over the third variable vanish in these formulae.

3.3. *Scalar product of two coherent states*

The projection of one coherent state $|\gamma'\rangle$ upon another one $|\gamma\rangle$ is also derived via the group multiplication formulae (2.20)

$$\begin{aligned}
\langle\gamma'|\gamma\rangle &= \langle\mu|g^\dagger[\gamma']g[\gamma]|\mu\rangle \\
&= \langle\mu|g[\beta'; 0, 0]g[\gamma; 0, 0]|\mu\rangle \\
&= \langle\mu|g[\tilde{\gamma}; \tilde{\psi}_1, \tilde{\psi}_2]|\mu\rangle
\end{aligned} \tag{3.11}$$

where $\tilde{\gamma}$ is connected to β and γ by (2.21). This leads to

$$\langle\gamma'|\gamma\rangle = \left(\frac{1 + \gamma_1'^*\gamma_1 + \gamma_3'^*\gamma_3}{\sqrt{f_1 f_1'}}\right)^{\lambda_1}\left(\frac{1 + \gamma_2'^*\gamma_2 + (\gamma_3'^* - \gamma_1'^*\gamma_2'^*)(\gamma_3 - \gamma_1\gamma_2)}{\sqrt{f_2 f_2'}}\right)^{\lambda_2}. \tag{3.12}$$

3.4. Resolution of unity

Coherent states form an overcomplete basis of the Hilbert space $\mathcal{H}_{[\lambda_1,\lambda_2]}$. This shows up in the fact that there exists a resolution of unity which is given by

$$\begin{aligned}\mathbb{I} &= \dim \mathcal{H}_{[\lambda_1,\lambda_2]} \int \mu |\gamma\rangle\langle\gamma| \\ &= \frac{(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)}{\pi^3} \int \mathrm{d}^2\gamma_1\,\mathrm{d}^2\gamma_2\,\mathrm{d}^2\gamma_3 \frac{1}{f_1^2 f_2^2} |\gamma\rangle\langle\gamma| \end{aligned} \tag{3.13}$$

for the non-degenerate representations. The measure μ appearing in (3.13) is the Haar measure on SU_3 given in section 2.6. The integration over the angles $\psi_{1,2}$ is carried out in the last line of (3.13) (that is the integration over the subgroup that leaves the highest state vector $|\mu\rangle$ invariant; one may call this the isotropy subgroup). For the degenerate representations we have to integrate over the full isotropy subgroup which is now four dimensional. This leads to

$$\begin{aligned}\mathbb{I} &= \frac{(\lambda_1+1)(\lambda_1+2)}{\pi^2} \int \mathrm{d}^2\gamma_1\,\mathrm{d}^2\gamma_3 \frac{1}{f_1^3} |\gamma\rangle\langle\gamma| \qquad \text{for} \quad \lambda_2 = 0 \\ \mathbb{I} &= \frac{(\lambda_2+1)(\lambda_2+2)}{\pi^2} \int \mathrm{d}^2\gamma_2\,\mathrm{d}^2\gamma_3 \frac{1}{(f_2|_{\gamma_1=0})^3} |\gamma\rangle\langle\gamma| \qquad \text{for} \quad \lambda_1 = 0. \end{aligned} \tag{3.14}$$

The proof of (3.13) and (3.14) is based on the Schur's lemma and $\mathrm{tr}\,\mathbb{I} = \dim \mathcal{H}_{[\lambda_1,\lambda_2]}$.

3.5. Expectation values of elements of the $\boldsymbol{sl}_3(\mathbb{C})$ Lie algebra and its universal enveloping algebra in the coherent states

If $\mathcal{O}$ is an operator in the universal enveloping algebra of $\boldsymbol{sl}_3(\mathbb{C})$ ($\mathcal{O}$ is a sum of products of generators of $\boldsymbol{sl}_3(\mathbb{C})$ of arbitrary order) then the expectation value of $\mathcal{O}S_{ij}$ in the coherent states may be expressed by the expectation value of $\mathcal{O}$,

$$\begin{aligned}
\langle\gamma|\mathcal{O}S_{32}|\gamma\rangle &= \left[\frac{\lambda_1}{f_1}\gamma_1\gamma_3^* + \frac{\lambda_2}{f_2}\gamma_2^* + \frac{\partial}{\partial\gamma_2} + \gamma_1\frac{\partial}{\partial\gamma_3}\right]\langle\gamma|\mathcal{O}|\gamma\rangle \\
\langle\gamma|\mathcal{O}S_{31}|\gamma\rangle &= \left[\frac{\lambda_1}{f_1}\gamma_3^* + \frac{\lambda_2}{f_2}(\gamma_3^* - \gamma_1^*\gamma_2^*) + \frac{\partial}{\partial\gamma_3}\right]\langle\gamma|\mathcal{O}|\gamma\rangle \\
\langle\gamma|\mathcal{O}S_{21}|\gamma\rangle &= \left[\frac{\lambda_1}{f_1}\gamma_1^* - \frac{\lambda_2}{f_2}\gamma_2(\gamma_3^* - \gamma_1^*\gamma_2^*) + \frac{\partial}{\partial\gamma_1}\right]\langle\gamma|\mathcal{O}|\gamma\rangle \\
\langle\gamma|\mathcal{O}S_{23}|\gamma\rangle &= \left[\frac{\lambda_1}{f_1}\gamma_1^*\gamma_3 + \frac{\lambda_2}{f_2}\gamma_2 + \gamma_3\frac{\partial}{\partial\gamma_1} - \gamma_2^2\frac{\partial}{\partial\gamma_2}\right]\langle\gamma|\mathcal{O}|\gamma\rangle \\
\langle\gamma|\mathcal{O}S_{13}|\gamma\rangle &= \left[\frac{\lambda_1}{f_1}\gamma_3 + \frac{\lambda_2}{f_2}(\gamma_3 - \gamma_1\gamma_2) - \gamma_1\gamma_3\frac{\partial}{\partial\gamma_1} - \gamma_2(\gamma_3 - \gamma_1\gamma_2)\frac{\partial}{\partial\gamma_2} - \gamma_3^2\frac{\partial}{\partial\gamma_3}\right]\langle\gamma|\mathcal{O}|\gamma\rangle \\
\langle\gamma|\mathcal{O}S_{12}|\gamma\rangle &= \left[\frac{\lambda_1}{f_1}\gamma_1 - \frac{\lambda_2}{f_2}\gamma_2^*(\gamma_3 - \gamma_1\gamma_2) - \gamma_1^2\frac{\partial}{\partial\gamma_1} - (\gamma_3 - \gamma_1\gamma_2)\frac{\partial}{\partial\gamma_2} - \gamma_1\gamma_3\frac{\partial}{\partial\gamma_3}\right]\langle\gamma|\mathcal{O}|\gamma\rangle
\end{aligned} \tag{3.15}$$

$$
\begin{aligned}
\langle\gamma|\mathcal{O}H_1|\gamma\rangle &= \left[\frac{\lambda_1}{2f_1}\left(1-|\gamma_1|^2\right)+\frac{\lambda_2}{2f_2}\left(|\gamma_2|^2-|\gamma_3-\gamma_1\gamma_2|^2\right)\right.\\
&\quad\left.-\gamma_1\frac{\partial}{\partial\gamma_1}+\frac{1}{2}\gamma_2\frac{\partial}{\partial\gamma_2}-\frac{1}{2}\gamma_3\frac{\partial}{\partial\gamma_3}\right]\langle\gamma|\mathcal{O}|\gamma\rangle\\
\langle\gamma|\mathcal{O}H_2|\gamma\rangle &= \left[\frac{\lambda_1}{2f_1}\left(|\gamma_1|^2-|\gamma_3|^2\right)+\frac{\lambda_2}{2f_2}\left(1-|\gamma_2|^2\right)\right.\\
&\quad\left.+\frac{1}{2}\gamma_1\frac{\partial}{\partial\gamma_1}-\gamma_2\frac{\partial}{\partial\gamma_2}-\frac{1}{2}\gamma_3\frac{\partial}{\partial\gamma_3}\right]\langle\gamma|\mathcal{O}|\gamma\rangle
\end{aligned}
$$

which can be calculated using the action of $\boldsymbol{sl}_3(\mathbb{C})$ generators on a coherent state (3.10). Using the above formulae we define differential operators ∂_{ij} via

$$\langle\gamma|\mathcal{O}S_{ij}|\gamma\rangle = \langle\gamma|S_{ij}|\gamma\rangle\langle\gamma|\mathcal{O}|\gamma\rangle + \partial_{ij}\langle\gamma|\mathcal{O}|\gamma\rangle. \tag{3.16}$$

Expectation values of the kind $\langle\gamma|S_{ij}\mathcal{O}|\gamma\rangle$ are found from

$$\langle\gamma|S_{ij}\mathcal{O}|\gamma\rangle = \langle\gamma|\mathcal{O}^\dagger S_{ji}|\gamma\rangle^*. \tag{3.17}$$

Setting $\mathcal{O}=\mathbb{I}$ in (3.15) gives the expectation values of the generators of $\boldsymbol{sl}_3(\mathbb{C})$ in coherent states

$$
\begin{aligned}
\langle\gamma|S_{32}|\gamma\rangle &= \frac{\lambda_1}{f_1}\gamma_1\gamma_3^*+\frac{\lambda_2}{f_2}\gamma_2^*\\
\langle\gamma|S_{31}|\gamma\rangle &= \frac{\lambda_1}{f_1}\gamma_3^*+\frac{\lambda_2}{f_2}(\gamma_3^*-\gamma_1^*\gamma_2^*)\\
\langle\gamma|S_{21}|\gamma\rangle &= \frac{\lambda_1}{f_1}\gamma_1^*-\frac{\lambda_2}{f_2}\gamma_2(\gamma_3^*-\gamma_1^*\gamma_2^*)\\
\langle\gamma|S_{23}|\gamma\rangle &= \frac{\lambda_1}{f_1}\gamma_1^*\gamma_3+\frac{\lambda_2}{f_2}\gamma_2\\
\langle\gamma|S_{13}|\gamma\rangle &= \frac{\lambda_1}{f_1}\gamma_3+\frac{\lambda_2}{f_2}(\gamma_3-\gamma_1\gamma_2)\\
\langle\gamma|S_{12}|\gamma\rangle &= \frac{\lambda_1}{f_1}\gamma_1-\frac{\lambda_2}{f_2}\gamma_2^*(\gamma_3-\gamma_1\gamma_2)\\
\langle\gamma|H_1|\gamma\rangle &= \frac{\lambda_1}{2f_1}\left(1-|\gamma_1|^2\right)+\frac{\lambda_2}{2f_2}\left(|\gamma_2|^2-|\gamma_3-\gamma_1\gamma_2|^2\right)\\
\langle\gamma|H_2|\gamma\rangle &= \frac{\lambda_1}{2f_1}\left(|\gamma_1|^2-|\gamma_3|^2\right)+\frac{\lambda_2}{2f_2}\left(1-|\gamma_2|^2\right).
\end{aligned}
\tag{3.18}
$$

Expectation values of squared generators are easily calculated as well using (3.17).

3.6. The SU_3-orbits on $\mathbb{P}\mathcal{H}_{[\lambda_1,\lambda_2]}$

Usually a quantum-mechanical state is described by a vector in a Hilbert space $\mathcal{H}$ appropriate to the system. However in a Hilbert space there are many vectors describing the same state. One may overcome this degeneracy by defining quantum mechanics on the projective space $\mathbb{P}\mathcal{H}$, the space of all rays in the Hilbert space $\mathcal{H}$. If $|\psi\rangle$ is a non-vanishing vector in $\mathcal{H}$ we write

$$[|\psi\rangle] := \{|\chi\rangle\in\mathcal{H}\colon |\chi\rangle = c|\psi\rangle \text{ for some } c\in\mathbb{C}\}. \tag{3.19}$$

Thus $[|\psi\rangle]$ is an equivalence class of vectors in $\mathcal{H}\backslash\{0\}$ and the projective space is the space of equivalence classes

$$\mathbb{P}\mathcal{H} = \mathcal{H}\backslash\{0\}/\sim \tag{3.20}$$

with dimension $\dim_{\mathbb{C}} \mathbb{P}\mathcal{H} = \dim_{\mathbb{C}}\mathcal{H} - 1$. If a group G acts on the Hilbert space $\mathcal{H}$ via a representation a group homomorphism on the projective space is induced

$$g[|\psi\rangle] = [g|\psi\rangle] \qquad \text{for} \quad g \in G. \tag{3.21}$$

Now an orbit $\mathcal{O}$ of the group G through a vector $|\psi\rangle \in \mathcal{H}$ or through a ray (or a state) $[|\psi\rangle] \in \mathbb{P}\mathcal{H}$ may be defined by

$$\begin{aligned} &\mathcal{O}_G(|\psi\rangle) = \{|\chi\rangle \in \mathcal{H}\colon |\chi\rangle = g|\psi\rangle \text{ for some } g \in G\} \equiv G|\psi\rangle \subset \mathcal{H} \\ &\mathcal{O}_G([|\psi\rangle]) = \{[|\chi\rangle] \in \mathbb{P}\mathcal{H}\colon [|\chi\rangle] = g[|\psi\rangle] \text{ for some } g \in G\} \equiv G[|\psi\rangle] \subset \mathbb{P}\mathcal{H}. \end{aligned} \tag{3.22}$$

We have defined a coherent state vector as an element of an $SU_3/I_{[\lambda_1,\lambda_2]}$-orbit through the highest weight vector $|\mu\rangle \in \mathcal{H}_{[\lambda_1,\lambda_2]}$. Here $I_{[\lambda_1,\lambda_2]}$ denotes the so-called isotropy subgroup of the highest weight defined by

$$I_{[\lambda_1,\lambda_2]} = \left\{g \in SU_3\colon g|\mu\rangle_{[\lambda_1,\lambda_2]} = \mathrm{e}^{\mathrm{i}t}|\mu\rangle_{[\lambda_1,\lambda_2]} \text{ for some } t \in \mathbb{R}\right\}. \tag{3.23}$$

For the non-degenerate representations we have

$$I_{[\lambda_1,\lambda_2]} = U_1 \times U_1 \qquad \text{for} \quad \lambda_1, \lambda_2 \neq 0 \tag{3.24}$$

and in the degenerate case

$$I_{[\lambda_1,\lambda_2]} = SU_2 \times U_1 \qquad \text{for} \quad \lambda_1 = 0 \quad \text{or} \quad \lambda_2 = 0. \tag{3.25}$$

Topologically the coherent states are equivalent to the quotient of SU_3 and the isotropy subgroup

$$\begin{aligned} &\mathcal{O}_{SU_3/U_1\times U_1}(|\mu\rangle) \simeq \mathcal{O}_{SU_3}([|\mu\rangle]) \simeq SU_3/U_1 \times U_1 \qquad \text{for} \quad \lambda_1, \lambda_2 \neq 0 \\ &\mathcal{O}_{SU_3/SU_2\times U_1}(|\mu\rangle) \simeq \mathcal{O}_{SU_3}([|\mu\rangle]) \simeq SU_3/SU_2 \times U_1 \qquad \text{for} \quad \lambda_1 = 0 \quad \text{or} \quad \lambda_2 = 0. \end{aligned} \tag{3.26}$$

As SU_3 is compact one may show that so is this quotient orbit. In the projective space $\mathbb{P}\mathcal{H}_{[\lambda_1,\lambda_2]}$ the coherent states form an SU_3-orbit $\mathcal{O}_{SU_3}([|\mu\rangle_{[\lambda_1,\lambda_2]}])$. This orbit in $\mathbb{P}\mathcal{H}_{[\lambda_1,\lambda_2]}$ is uniquely determined by one of the following properties [24]:

- it is both a $SL_3(\mathbb{C})$-orbit and a SU_3-orbit;
- it is a unique closed $SL_3(\mathbb{C})$-orbit;
- it is a unique complex SU_3-orbit.

The orbits $\mathcal{O}_{SU_3/I_{[\lambda_1,\lambda_2]}}(|\mu\rangle_{[\lambda_1,\lambda_2]}) \subset \mathcal{H}_{[\lambda_1,\lambda_2]}$ and $\mathcal{O}_{SU_3}([|\mu\rangle_{[\lambda_1,\lambda_2]}]) \subset \mathbb{P}\mathcal{H}_{[\lambda_1,\lambda_2]}$ are easily seen to be isomorphic. The complex structure is easily recovered if we recall our former definition which used $\mathbb{C}^3$ and $\mathbb{C}^2$ to parametrize the coherent states.

4. The symplectic structure on coherent states and the classical limit

The coherent states in $\mathbb{P}\mathcal{H}_{[\lambda_1,\lambda_2]}$ also possess a symplectic structure that is invariant under the group action. This makes this orbit very interesting for defining a classical limit for a quantum-mechanical system which is an inverse procedure to geometrical quantization (the mathematical and physical literature is very large, so we mention only a few important contributions [25–30], most books and papers cited elsewhere in our paper also deal with this subject from some point of view).

4.1. The symplectic structure on a Hilbert space

On any complex Hilbert space $\mathcal{H} \equiv \mathbb{C}^n$ one may canonically define a symplectic structure $\omega_{\mathcal{H}}$ using the canonical scalar product [31, 32]

$$\omega_{\mathcal{H}}(|\phi_1\rangle, |\phi_2\rangle) = 2\hbar\,\mathrm{Im}\{\langle\phi_1|\phi_2\rangle\} = \hbar(\langle\phi_1|\phi_2\rangle - \langle\phi_2|\phi_1\rangle) \qquad \text{for} \quad |\phi_1\rangle, |\phi_2\rangle \in \boldsymbol{T}_{|x\rangle}\mathcal{H}. \tag{4.1}$$

We have identified the Hilbert space $\mathcal{H}$ with its tangent space $\boldsymbol{T}_{|x\rangle}\mathcal{H}$ at $|x\rangle \in \mathcal{H}$ here. Thus every Hilbert space of complex dimension $\dim_{\mathbb{C}} \mathcal{H} = n$ is also a linear symplectic space of real dimension $\dim_{\mathbb{R}} \mathcal{H} = 2n$. In quantum mechanics the Hamiltonian $H \in \mathrm{i}\boldsymbol{u}_n$ generates a unitary action on the Hilbert space. As unitary action leaves the scalar product invariant, the quantum-mechanical dynamics leaves the symplectic structure invariant. Formally quantum mechanics is a special case of Hamiltonian mechanics. More explicitly we may introduce an orthonormal basis $|k\rangle$, $k = 1, \ldots, n$ and write a vector $|\psi\rangle = \sum_{k=1}^n z_k|k\rangle = \sum_{k=1}^n (q_k + \mathrm{i}p_k)|k\rangle$, $q_k, p_k \in \mathbb{R}$, $z_k \in \mathbb{C}$ with real coordinates q_k and p_k. In these coordinates the symplectic 2-form has the canonical form

$$\omega_{\mathcal{H}} = \mathrm{i}\hbar \sum_{k=1}^n \mathrm{d}z_k \wedge \mathrm{d}z_k^* = 2\hbar \sum_{k=1}^n \mathrm{d}q_k \wedge \mathrm{d}p_k. \tag{4.2}$$

Define the Hamilton function

$$\begin{aligned} h:&\quad \mathcal{H} \to \mathbb{R} \\ |\psi\rangle \to h(|\psi\rangle) &= \langle\psi|H|\psi\rangle = \sum_{k,l=1}^n \langle k|H|l\rangle(q_k - \mathrm{i}p_k)(q_l + \mathrm{i}p_l). \end{aligned} \tag{4.3}$$

The Schrödinger equation has now the form of Hamilton's equations with respect to the symplectic 2-form (4.2) [32]

$$\begin{gathered} \mathrm{i}\hbar|\dot\psi\rangle = H|\psi\rangle \\ \Longleftrightarrow \\ \dot q_k = \frac{1}{2\hbar}\frac{\partial h}{\partial p_k} \qquad \text{and} \qquad \dot p_k = -\frac{1}{2\hbar}\frac{\partial h}{\partial q_k}. \end{gathered} \tag{4.4}$$

As stated before we may define quantum mechanics as well in the projective space $\mathbb{P}\mathcal{H}$. The projective space inherits a Hermitian and a symplectic structure from its underlying Hilbert space. Quantum dynamics on the projective space may then be written again in the form of Hamilton's equations. We shall give the symplectic structure explicitly in affine coordinates. The orthonormal basis $\{|k\rangle\}_{k=1,\ldots,n}$ of the Hilbert space $\mathcal{H}$ induces n local so-called affine charts on open sets $U_k = \{[|\psi\rangle] \in \mathbb{P}\mathcal{H}\colon \langle k|\psi\rangle \neq 0\} \subset \mathbb{P}\mathcal{H}$, $k = 1, \ldots, n$. Affine coordinates on for $[|\psi\rangle] \in U_n$ where $|\psi\rangle = \sum_{k=1}^n z_k|k\rangle$ are given by

$$w_l = \frac{z_l}{z_k} \qquad l = 1, \ldots, n \quad w_k = 1 \tag{4.5}$$

one usually writes $[|\psi\rangle] = [z_1 : z_2 : \ldots : z_n] = [w_1 : \ldots : w_k = 1 : \ldots : w_n] = [w]$ to define a ray in U_k. The symplectic 2-form in these affine coordinates is given explicitly by [31]

$$\omega_{\mathbb{P}\mathcal{H}}|_{U_k} = \mathrm{i}\hbar \sum_{r,s=1;r,s\neq k}^n \frac{\partial^2 \log \mathcal{N}_k}{\partial w_r^* \, \partial w_s} \mathrm{d}w_r \wedge \mathrm{d}w_s^* \tag{4.6}$$

with

$$\mathcal{N}_k = \sum_{t=1}^n |w_t|^2 \qquad \text{with} \quad |w_k|^2 = 1. \tag{4.7}$$

If we embed U_k in the Hilbert space by

$$P_k:\quad U_k \to \mathcal{H}$$
$$P_k[w] = \sum_{l=1}^{n} w_l |l\rangle \equiv |w\rangle_k \tag{4.8}$$

we may write $\mathcal{N}_k$ as the squared norm

$$\mathcal{N}_k = |P_k[w]|^2 = \langle w|w\rangle_k. \tag{4.9}$$

We shall call the function $\mathcal{N}_k$ a generating function of the symplectic 2-form. In the next section we shall sketch how the symplectic structure on $\mathcal{H}$ or $\mathbb{P}\mathcal{H}$ gives a SU_3-invariant symplectic structure on the submanifold of SU_3-coherent states if an irreducible unitary representation of SU_3 acts on $\mathcal{H}$. Note that for $\mathcal{H} = \mathbb{C}^n$ we may define SU_n-coherent states. Now every normalized vector in $\mathbb{C}^n$ is a SU_n-coherent state up to a phase factor, in fact the manifold of SU_n-coherent states on $\mathbb{C}^n$ may be identified with the projective space $\mathbb{CP}^{n-1} \equiv \mathbb{PC}^n$. Now the symplectic structure defined on $\mathbb{CP}^{n-1}$ is clearly group invariant with respect to SU_n.

4.2. The symplectic structure on SU_3 coherent states

Let us now consider the Hilbert space $\mathcal{H}_{[\lambda_1,\lambda_2]}$ of an irreducible SU_3-representation. The coherent states $\mathcal{O}_{SU_3}([|\mu\rangle_{[\lambda_1,\lambda_2]}])$ form an SU_3-invariant submanifold of $\mathbb{P}\mathcal{H}_{[\lambda_1,\lambda_2]}$. It is therefore possible to reduce the symplectic 2-form (4.6) on the projective space to a SU_3-invariant 2-form on the SU_3-orbit of coherent states. Let us assume as before that we have an orthonormal basis $|k\rangle$, $k = 1, \ldots, n$ in $\mathcal{H}_{[\lambda_1,\lambda_2]}$ where $n = \dim_{\mathbb{C}} \mathcal{H}_{[\lambda_1,\lambda_2]}$ and that the highest weight vector in this basis $|\mu\rangle = |1\rangle$. If we now embed coherent states in $U_1 \subset \mathbb{P}\mathcal{H}_{[\lambda_1,\lambda_2]}$ in the Hilbert space $\mathcal{H}_{[\lambda_1,\lambda_2]}$ via P_1 (see equation (4.8)) we see that we arrive at an unnormalized coherent state vector in $\mathcal{H}_{[\lambda_1,\lambda_2]}$

$$P_1[|\gamma\rangle] = \|\gamma\rangle. \tag{4.10}$$

We have seen before that the generating function $\mathcal{N}_1$ for the symplectic 2-form on the projective space is given by the norm of the vectors in $\mathcal{H}_{[\lambda_1,\lambda_2]}$ associated to rays by the embedding P_1. All we have to do is to restrict this embedding to coherent states

$$\omega_{SU_3} = \mathrm{i}\hbar \sum_{i,j=1}^{3} \omega_{ij}\, \mathrm{d}\gamma_i \wedge \mathrm{d}\gamma_j^*$$
$$\omega_{ij} = \frac{\partial^2 \mathcal{F}}{\partial\gamma_i\, \partial\gamma_j^*} \tag{4.11}$$

$$\mathcal{F} = \log\left(f_1^{\lambda_1} f_2^{\lambda_2}\right) = \log\langle\gamma\|\gamma\rangle. \tag{4.12}$$

For the degenerate representations we have to sum over two indices only. Explicitly the coefficients ω_{ij} read

$$\omega_{11} = \frac{\lambda_1}{f_1^2}\left(1 + |\gamma_3|^2\right) + \frac{\lambda_2}{f_2^2}|\gamma_2|^2\left(1 + |\gamma_2|^2\right)$$
$$\omega_{12} = \frac{\lambda_2}{f_2^2}\gamma_2(\gamma_1^* + \gamma_2\gamma_3^*)$$
$$\omega_{13} = -\frac{\lambda_1}{f_1^2}\gamma_1^*\gamma_3 - \frac{\lambda_2}{f_2^2}\gamma_2\left(1 + |\gamma_2|^2\right)$$

$$
\begin{aligned}
\omega_{21} &= \frac{\lambda_2}{f_2^2}\gamma_2^*(\gamma_1+\gamma_2^*\gamma_3)\\
\omega_{22} &= \frac{\lambda_2}{f_2^2}f_1\\
\omega_{23} &= -\frac{\lambda_2}{f_2^2}(\gamma_1+\gamma_2^*\gamma_3)\\
\omega_{31} &= -\frac{\lambda_1}{f_1^2}\gamma_1\gamma_3^* - \frac{\lambda_2}{f_2^2}\gamma_2^*\left(1+|\gamma_2|^2\right)\\
\omega_{32} &= -\frac{\lambda_2}{f_2^2}(\gamma_1^*+\gamma_2\gamma_3^*)\\
\omega_{33} &= \frac{\lambda_1}{f_1^2}\left(1+|\gamma_1|^2\right)+\frac{\lambda_2}{f_2^2}\left(1+|\gamma_2|^2\right).
\end{aligned}
\tag{4.13}
$$

The symplectic form may also be written in the form

$$
\begin{aligned}
\omega_{SU_3} = {}& \mathrm{i}\hbar\lambda_1\left[\mathrm{d}\frac{\gamma_1}{\sqrt{f_1}}\wedge\mathrm{d}\frac{\gamma_1^*}{\sqrt{f_1}}+\mathrm{d}\frac{\gamma_3}{\sqrt{f_1}}\wedge\mathrm{d}\frac{\gamma_3^*}{\sqrt{f_1}}\right]\\
&+\mathrm{i}\hbar\lambda_2\left[\mathrm{d}\frac{\gamma_2}{\sqrt{f_2}}\wedge\mathrm{d}\frac{\gamma_2^*}{\sqrt{f_2}}+\mathrm{d}\frac{\gamma_3-\gamma_1\gamma_2}{\sqrt{f_2}}\wedge\mathrm{d}\frac{\gamma_3^*-\gamma_1^*\gamma_2^*}{\sqrt{f_2}}\right]
\end{aligned}
\tag{4.14}
$$

which shows how it becomes degenerate in the limit $\lambda_i \to 0$. For $\lambda_i = 0$ one can also read from (4.11) the appropriate canonical variables in both degenerates cases. We shall deal with canonical variables in section 4.4.

The invariant 1-forms on the group $SU(3)$ derived in section 2.6 induce $SU_3/I_{[\lambda_1,\lambda_2]}$ invariant 1-forms on the orbits $\mathcal{O}_{SU_3/I_{[\lambda_1,\lambda_2]}}(|\mu\rangle)$ and $\mathcal{O}_{SU_3}([|\mu\rangle])$ as well. In the non-degenerate case there are six independent invariant 1-forms. As we used the same coordinates for the parametrization of the group SU_3, for the quotient $SU_3/I_{[\lambda_1,\lambda_2]}$ and the coherent states these induced 1-forms are just given by restricting the 1-forms α_{γ_i} in (2.24) appropriately

$$
\tilde{\alpha}_{\gamma_i} = \alpha_{\gamma_i}|_{\psi_1=0,\psi_2=0}. \tag{4.15}
$$

In terms of these invariant 1-forms the symplectic 2-form ω_{SU_3} is diagonal

$$
\omega_{SU_3} = \mathrm{i}\hbar[\lambda_1\tilde{\alpha}_{\gamma_1}\wedge\tilde{\alpha}_{\gamma_1^*}+\lambda_2\tilde{\alpha}_{\gamma_2}\wedge\tilde{\alpha}_{\gamma_2^*}+(\lambda_1+\lambda_2)\tilde{\alpha}_{\gamma_3}\wedge\tilde{\alpha}_{\gamma_3^*}]. \tag{4.16}
$$

In the resolution of unity (3.13) the Haar measure is integrated over the isotropy subgroup $I_{[\lambda_1,\lambda_2]}$, In the non-degenerate case this partly integrated measure is related to the third power of the symplectic 2-form

$$
\int_{U_1\times U_1}\mu = c\,\omega_{SU_3}^3 \tag{4.17}
$$

with a constant c which may be deduced from (3.13) using

$$
\det\omega_{ij} = \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2)}{f_1^2 f_2^2}. \tag{4.18}
$$

With the given symplectic 2-form we may construct Poisson brackets for complex-valued functions on the orbits of coherent states. The Poisson bracket for any two functions $f(\gamma), g(\gamma) \in C^1(\mathcal{O}_{SU_3}([|\mu\rangle]))$ on the orbit of coherent states is determined by

$$
\{f,g\} = \frac{\mathrm{i}}{\hbar}\sum_{k,l=1}^{3}\omega^{kl}\left(\frac{\partial f}{\partial\gamma_l}\frac{\partial g}{\partial\gamma_k^*}-\frac{\partial f}{\partial\gamma_k^*}\frac{\partial g}{\partial\gamma_l}\right) \tag{4.19}
$$

where the coefficient matrix ω^{ij} is just the inverse matrix of ω_{ij}

$$\omega^{ik}\omega_{kj} = \delta^i_j. \tag{4.20}$$

In the non-degenerate case this matrix is given explicitly by

$$\begin{aligned}
\omega^{11} &= \frac{f_1}{(\lambda_1+\lambda_2)}\left(\left(1+|\gamma_1|^2\right)+\frac{\lambda_2}{\lambda_1}\frac{f_1}{f_2}\right)\\
\omega^{12} &= \frac{1}{(\lambda_1+\lambda_2)}(\gamma_1^*+\gamma_2\gamma_3^*)\left(\gamma_1^*\gamma_3-\gamma_2-\gamma_2|\gamma_1|^2\right)\\
\omega^{13} &= \frac{f_1}{(\lambda_1+\lambda_2)}\left(\gamma_1^*\gamma_3+\frac{\lambda_2}{\lambda_1}\frac{f_1}{f_2}\gamma_2\right)\\
\omega^{21} &= \frac{1}{(\lambda_1+\lambda_2)}(\gamma_1+\gamma_2^*\gamma_3)\left(\gamma_1\gamma_3^*-\gamma_2^*-\gamma_2^*|\gamma_1|^2\right)\\
\omega^{22} &= \frac{f_2}{(\lambda_1+\lambda_2)}\left(\left(1+|\gamma_2|^2\right)+\frac{\lambda_1}{\lambda_2}\frac{f_2}{f_1}\right)\\
\omega^{23} &= \frac{1}{(\lambda_1+\lambda_2)}(\gamma_1+\gamma_2^*\gamma_3)\left(1+|\gamma_3|^2-\gamma_1^*\gamma_2^*\gamma_3\right)\\
\omega^{31} &= \frac{f_1}{(\lambda_1+\lambda_2)}\left(\gamma_1\gamma_3^*+\frac{\lambda_2}{\lambda_1}\frac{f_1}{f_2}\gamma_2^*\right)\\
\omega^{32} &= \frac{1}{(\lambda_1+\lambda_2)}(\gamma_1^*+\gamma_2\gamma_3^*)\left(1+|\gamma_3|^2-\gamma_1\gamma_2\gamma_3^*\right)\\
\omega^{33} &= \frac{f_1}{(\lambda_1+\lambda_2)}\left(\left(1+|\gamma_3|^2\right)+\frac{\lambda_2}{\lambda_1}\frac{f_1}{f_2}|\gamma_2|^2\right).
\end{aligned} \tag{4.21}$$

In the degenerate case only the appropriate 2×2 submatrix has to be taken (with the additional settings given above). These Poisson brackets are connected to the commutators of the Lie algebra $\boldsymbol{su}_3$ by

$$\mathrm{i}\langle\gamma|[A,B]|\gamma\rangle = \hbar\{\langle\gamma|A|\gamma\rangle,\langle\gamma|B|\gamma\rangle\} \qquad \text{for} \quad A,B\in\boldsymbol{su}_3. \tag{4.22}$$

4.3. Coherent states and orbits of the coadjoint representation

Above we have sketched how a SU_3-invariant symplectic structure on coherent states is defined by restriction of the symplectic 2-form in the Hilbert space to the submanifold of coherent states. There is another very instructive way to look at this symplectic structure. Here we shall sketch a one-to-one map from the orbit of coherent states $\mathcal{O}_{SU_3}([|\mu\rangle])$ to $\boldsymbol{su}_3^*$ the dual of the Lie algebra $\boldsymbol{su}_3$. This so-called momentum map is surjective on submanifolds of $\boldsymbol{su}_3^*$ called orbits of the coadjoint representation.

Before defining the momentum map we shall shortly summarize some basic facts about the coadjoint representations of a group G and about the symplectic structure on $\boldsymbol{g}^*$ (the dual of the Lie algebra $\boldsymbol{g} = \boldsymbol{T}|_e G$). We assume here that the Lie algebra $\boldsymbol{g}$ has finite dimension (more general results may be found in [18, 31, 32]). The adjoint representation is a homomorphism of the group G on the automorphisms of its algebra $\mathrm{Ad}: G \to \mathrm{Aut}(\boldsymbol{g})$. If $g\in G$ and $X\in\boldsymbol{g}$ one defines

$$\begin{aligned}
&g\in G \longmapsto \mathrm{Ad}_g : \boldsymbol{g}\to\boldsymbol{g}\\
&\mathrm{Ad}_g\, X = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g\circ \mathrm{e}^{Xt}\circ g^{-1} \qquad \text{for} \quad X\in\boldsymbol{g}.
\end{aligned} \tag{4.23}$$

If G is a matrix group this reduces to

$$\mathrm{Ad}_g X = gXg^{-1} \tag{4.24}$$

where the product is calculated by matrix multiplication. The coadjoint representation $\mathrm{Ad}^* : G \to \boldsymbol{g}^*$ is defined via the pairing $(\xi, X) \in \mathbb{C}$ for $\xi \in \boldsymbol{g}^*$ and $X \in \boldsymbol{g}$

$$\begin{aligned} & g \in G \longmapsto \mathrm{Ad}^*_g : \boldsymbol{g}^* \to \boldsymbol{g}^* \\ & \mathrm{Ad}^*_g \xi : \quad \boldsymbol{g} \to \mathbb{C} \qquad \text{for} \quad \xi \in \boldsymbol{g}^* \qquad \text{such that} \\ & (\mathrm{Ad}^*_g \xi, X) = (\xi, \mathrm{Ad}_{g^{-1}} X) \qquad \forall\, X \in \boldsymbol{g}. \end{aligned} \tag{4.25}$$

If G is a semisimple matrix group one may define a non-degenerate scalar product on the algebra $\boldsymbol{g}$ coinciding with the so-called Killing form $K : \boldsymbol{g} \times \boldsymbol{g} \to \mathbb{C}$. It is given by

$$K(Y, X) = \mathrm{tr}(Y^\dagger X) \qquad X, Y \in \boldsymbol{g} \tag{4.26}$$

where $Y^\dagger$ is the matrix Hermitian conjugate to Y. Using the Killing form we may identify $\boldsymbol{g}$ with $\boldsymbol{g}^*$ via $X \to K_X \equiv K(X, \cdot) : \boldsymbol{g} \to \mathbb{C}$. Now Ad^* may be given more explicitly

$$\mathrm{Ad}^*_g K_X = K_{g^{-1\dagger} X g^\dagger} \qquad \text{for} \quad X \in \boldsymbol{g} \to K_X \in \boldsymbol{g}^*. \tag{4.27}$$

If G is a matrix group of unitary matrices $g^\dagger = g^{-1}$ we have $\mathrm{Ad}^*_g K_X = K_{\mathrm{Ad}_g X}$. Using the Lie bracket on $\boldsymbol{g}$ one may construct a Poisson bracket for functions $f : \boldsymbol{g}^* \to \mathbb{C}$. The Poisson bracket is most easily defined for linear functionals. As $\boldsymbol{g}^{**} \equiv \boldsymbol{g}$ every $X \in \boldsymbol{g}$ defines such a linear functional

$$\begin{aligned} & f_X : \quad \boldsymbol{g}^* \to \mathbb{C} \qquad \text{for} \quad X \in \boldsymbol{g} \\ & f_X(\xi) = (\xi, X) \in \mathbb{C} \qquad \text{for} \quad \xi \in \boldsymbol{g}^*. \end{aligned} \tag{4.28}$$

For these linear functionals the Poisson bracket is defined by

$$\{f_X, f_Y\} = f_{[X,Y]}. \tag{4.29}$$

Using the Leibnitz rule one may easily construct the Poisson bracket for all functions on $\boldsymbol{g}^*$ that may be written as a series in the linear functionals [31, 32]. There is an action of the group G on functions $f : \boldsymbol{g}^* \to \mathbb{C}$

$$g \circ f(\xi) = f(\mathrm{Ad}^*_{g^{-1}} \xi) \tag{4.30}$$

and the Poisson bracket is invariant with respect to this action in the sense

$$g \circ \{f_1, f_2\} = \{g \circ f_1, g \circ f_2\}. \tag{4.31}$$

In general, there will be no closed non-degenerate (that is symplectic) 2-form corresponding to these Poisson brackets. In particular, there might be non-trivial so-called Casimir functions c_ν. These Casimir functions have (per definition) a vanishing Poisson bracket with any other function on $\boldsymbol{g}^*$. Any function that is invariant under the action of the group

$$g \circ c_\nu = c_\nu \tag{4.32}$$

will be a Casimir function. These invariant functions will be constant on the orbits of the coadjoint representation (or just coadjoint orbits) which are homogeneous submanifolds of $\boldsymbol{g}^*$ (homogeneous here means that any two elements are connected by a group action)

$$\mathcal{O}_G(\xi) = \{\zeta \in \boldsymbol{g}^* : \zeta = \mathrm{Ad}^*_g \xi \text{ for some } g \in G\} \equiv G \circ \xi \qquad \text{for} \quad \xi \in \boldsymbol{g}^*. \tag{4.33}$$

It is possible to restrict the Poisson brackets to functions on these coadjoint orbits and one may show that for many groups there is a non-degenerate closed 2-form connected to the Poisson bracket that makes them homogeneous symplectic manifolds.

Let us now go back to $G = SU_3$. We shall think of $\boldsymbol{su}_3$ and $\boldsymbol{su}_3^*$ as the set of all anti-Hermitian traceless 3×3 matrices (though we refer to them as the same set we keep on to distinguish them due to their different Lie structures). These matrices have purely imaginary eigenvalues and these eigenvalues are invariant under the (co)adjoint action. A nice way to think of a coadjoint orbit is the subset $\mathcal{O}_{SU_3}(\xi) \subset \boldsymbol{su}_3^*$ of all matrices with the same eigenvalues. In particular, there is a diagonal matrix $d \in \mathcal{O}_{SU_3}(\xi)$ in every coadjoint orbit such that the eigenvalues are ordered

$$d = \begin{pmatrix} \mathrm{i}d_{11} & 0 & 0 \\ 0 & \mathrm{i}d_{22} & 0 \\ 0 & 0 & \mathrm{i}d_{33} \end{pmatrix} \qquad \text{such that} \quad d_{ii} \in \mathbb{R} \quad d_{11} \geqslant d_{22} \geqslant d_{33}. \tag{4.34}$$

As $d_{11} + d_{22} + d_{33} = 0$ we may label each coadjoint orbit by two non-negative numbers

$$\Lambda_1 = d_{11} - d_{22} \qquad \Lambda_2 = d_{22} - d_{33} \tag{4.35}$$

and we shall denote the coadjoint orbit through the matrix d by $\mathcal{O}_{SU_3}[\Lambda_1, \Lambda_2] \equiv SU_3 d$. We may parametrize coadjoint orbits using the group parametrization of section 2.4 if we write $g[\gamma] \circ d \circ g[\gamma]^\dagger$ for an element of the coadjoint orbit. To make this parametrization unambiguous we have to divide out isotropy subgroups of the diagonal matrix d (that is the subgroup of all matrices $g \in SU_3$ such that $d = g \circ d \circ g^\dagger$). Instead of reducing the Poisson brackets from $\boldsymbol{su}_3^*$ to the coadjoint orbits we shall identify coadjoint orbits with the orbits of coherent states and show that the symplectic structure on coherent states now considered as a submanifold of $\boldsymbol{su}_3^*$ coincides with the Poisson brackets on $\boldsymbol{su}_3^*$. We shall identify the coadjoint orbit $\mathcal{O}_{SU_3}[\Lambda_1, \Lambda_2]$ with the orbit of coherent states $\mathcal{O}_{SU_3}([|\mu\rangle]) \subset \mathbb{P}\mathcal{H}_{[\lambda_1,\lambda_2]}$ for $\Lambda_1 = \hbar\lambda_1$, $\Lambda_2 = \hbar\lambda_2$ via the so-called momentum map†

$$\begin{aligned} &J: \quad \mathcal{O}_{SU_3}([|\mu\rangle]) \rightarrow \boldsymbol{su}_3^* \\ &[|\gamma\rangle] \rightarrow J([|\gamma\rangle]) \\ &J([|\gamma\rangle]) = \mathrm{i}\hbar \\ &\times \begin{pmatrix} \frac{4}{3}\langle\gamma|H_1|\gamma\rangle + \frac{2}{3}\langle\gamma|H_2|\gamma\rangle & \langle\gamma|S_{12}|\gamma\rangle & \langle\gamma|S_{13}|\gamma\rangle \\ \langle\gamma|S_{21}|\gamma\rangle & -\frac{2}{3}\langle\gamma|H_1|\gamma\rangle + \frac{2}{3}\langle\gamma|H_2|\gamma\rangle & \langle\gamma|S_{23}|\gamma\rangle \\ \langle\gamma|S_{31}|\gamma\rangle & \langle\gamma|S_{32}|\gamma\rangle & -\frac{2}{3}\langle\gamma|H_1|\gamma\rangle - \frac{4}{3}\langle\gamma|H_2|\gamma\rangle \end{pmatrix}. \end{aligned} \tag{4.36}$$

We shall write

$$s_{ij} = \hbar\langle\gamma|S_{ij} - \tfrac{1}{3}\delta_{ij}N|\gamma\rangle = -\mathrm{i}J([|\gamma\rangle])_{ij} \tag{4.37}$$

with $N = S_{11} + S_{22} + S_{33}$. For this momentum map we have

$$J(g[|\gamma\rangle]) = \mathrm{Ad}_g^*(J([|\gamma\rangle])) = g \circ J([|\gamma\rangle]) \circ g^\dagger \qquad \forall g \in SU_3 \tag{4.38}$$

and $J([|\mu\rangle]) = d$.

The expectation values $\mathrm{i}s_{ij}$ may now be considered as functions on the coadjoint orbit or as functions on the orbit of coherent states. We may construct from them eight independent real linear functionals on $\boldsymbol{su}_3^*$. We have defined the Poisson brackets on $\boldsymbol{su}_3^*$ in (4.29) using such linear functionals (they correspond to elements of $\boldsymbol{su}_3$). One may now check using (4.22) that the Poisson brackets on $\boldsymbol{su}_3^*$ fit to the symplectic structure on coherent states obtained by reduction from the canonical symplectic structure of the Hilbert space. Coadjoint orbits with the symplectic 2-form (4.11) may be considered as classical phase spaces for Hamiltonian dynamics.

† In general, one defines the momentum map as a linear map from a symplectic manifold to the dual of the Lie algebra of a group that acts as a Hamiltonian on the symplectic manifold, which is equivariant with respect to the action of the group on the manifold and the coadjoint action of the group on its dual algebra [32].

There are two functionally independent Casimir functions on $\boldsymbol{su}_3^*$ given by

$$c_1 = \sum_{k,l=1}^{3} s_{kl}s_{lk}$$
$$c_2 = \sum_{k,l,m=1}^{3} s_{kl}s_{lm}s_{mk}. \tag{4.39}$$

Using (4.30) we see that they are in fact constant on coadjoint orbits.

4.4. Canonical coordinates on orbits of the coadjoint representation

By Darboux's theorem we know that every symplectic 2-form ω on a manifold M may be locally brought to the form

$$\omega|_U = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i \tag{4.40}$$

by coordinate transformations. Such canonical coordinates are very useful in explicit investigations of dynamical systems. We shall give here a set of canonical coordinates on coadjoint orbits in the form of action-angle variables. First observe that in (4.11) we have used the generating function F to define the symplectic 2-form on coherent states (and via the momentum map on coadjoint orbits). It is easily seen that one may rewrite (4.11) as

$$\omega_{SU_3} = \mathrm{i}\hbar \sum_{k=1}^{3} \mathrm{d}\gamma_k \wedge \mathrm{d}\frac{\partial \mathcal{F}}{\partial \gamma_k}. \tag{4.41}$$

With

$$\xi_k = \mathrm{i}\hbar \frac{\partial \mathcal{F}}{\partial \gamma_k} \qquad k = 1, 2, 3 \tag{4.42}$$

the symplectic 2-form has the canonical form. However, γ and ξ are complex variables and it is not trivial to reduce them to real canonical variables as $\mathrm{d}\gamma_k \wedge \mathrm{d}\xi_k$ is not real for $k = 1, 2, 3$ (the whole symplectic 2-form ω_{SU_3} is real).

A canonical transformation to real coordinates may be found by observing that

$$I_1 = y = \tfrac{1}{3}(s_{11} + s_{22} - 2s_{33})$$
$$I_2 = t_3 = \tfrac{1}{2}(s_{11} - s_{22}) \tag{4.43}$$
$$I_3 = t = \sqrt{t_3^2 + s_{12}s_{21}}$$

have vanishing Poisson brackets

$$\{I_i, I_j\} = 0 \qquad \text{for} \quad i, j = 1, 2, 3. \tag{4.44}$$

These three real functions are the classical counterparts of well known quantum numbers in elementary particle physics. We shall adopt the language of elementary particle physics and call y the (classical limit of) hypercharge, t_3 the 3-component of isospin and t the (total) isospin. Expressed in the complex canonical coordinates these functions read

$$I_1 = y = \hbar\tfrac{1}{3}(\lambda_1 + 2\lambda_2) + \mathrm{i}\gamma_3\xi_3 + \mathrm{i}\gamma_2\xi_2$$
$$I_2 = t_3 = \tfrac{1}{2}\hbar\lambda_1 + \mathrm{i}\gamma_1\xi_1 - \mathrm{i}\tfrac{1}{2}\gamma_2\xi_2 + \mathrm{i}\tfrac{1}{2}\gamma_3\xi_3 \tag{4.45}$$
$$I_3 = t = \sqrt{\tfrac{1}{4}(\hbar\lambda_1 - \mathrm{i}\gamma_2\xi_2 + \mathrm{i}\gamma_3\xi_3)^2 + \gamma_3\xi_1\xi_2}.$$

The above equation can be solved for γ in terms of I and ξ†

$$\gamma_1 = -\frac{1}{\xi_1}\left(\frac{\xi_1\xi_2}{2\xi_3} + \mathrm{i}I_2 + \sqrt{\left(\frac{\xi_1\xi_2}{2\xi_3}\right)^2 + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)\frac{\xi_1\xi_2}{\xi_3} - I_3^2}\right)$$

$$\gamma_2 = -\frac{1}{\xi_2}\left(\frac{\xi_1\xi_2}{2\xi_3} + \mathrm{i}\left(\tfrac{1}{2}I_1 - \tfrac{1}{3}\hbar(\lambda_2-\lambda_1)\right) - \sqrt{\left(\frac{\xi_1\xi_2}{2\xi_3}\right)^2 + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)\frac{\xi_1\xi_2}{\xi_3} - I_3^2}\right) \tag{4.46}$$

$$\gamma_3 = \frac{1}{\xi_3}\left(\frac{\xi_1\xi_2}{2\xi_3} + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right) + \sqrt{\left(\frac{\xi_1\xi_2}{2\xi_3}\right)^2 + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)\frac{\xi_1\xi_2}{\xi_3} - I_3^2}\right).$$

The canonical conjugated functions are found with the help of a generating function $S(I,\xi)$ such that

$$\gamma_k(I,\xi) = \frac{\partial S(I,\xi)}{\partial \xi_k}. \tag{4.47}$$

Such a generating function is given by

$$\begin{aligned}
S(I,\xi) &= -\frac{\xi_1\xi_2}{2\xi_3} - \mathrm{i}I_2\log\xi_1 + \mathrm{i}\left(\tfrac{1}{3}\hbar(\lambda_2-\lambda_1) - \tfrac{1}{2}I_1\right)\log\xi_2 + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)\log\xi_3 \\
&\quad - \int \mathrm{d}z\,\frac{1}{z}\sqrt{z^2 + 2\mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)z - I_3^2}\,\Bigg|_{z=\xi_1\xi_2/2\xi_3} \\
&= -\frac{\xi_1\xi_2}{2\xi_3} - \mathrm{i}I_2\log\xi_1 + \mathrm{i}\left(\tfrac{1}{3}\hbar(\lambda_2-\lambda_1) - \tfrac{1}{2}I_1\right)\log\xi_2 \\
&\quad - \sqrt{\left(\frac{\xi_1\xi_2}{2\xi_3}\right)^2 + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)\frac{\xi_1\xi_2}{\xi_3} - I_3^2} \\
&\quad + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)\Bigg(1 + \log\xi_3 \\
&\quad - \log\Bigg[\mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right) + \frac{\xi_1\xi_2}{2\xi_3} \\
&\quad + \sqrt{\left(\frac{\xi_1\xi_2}{2\xi_3}\right)^2 + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)\frac{\xi_1\xi_2}{\xi_3} - I_3^2}\,\Bigg]\Bigg) \\
&\quad - \mathrm{i}I_3\log\Big[-I_3^2 + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)\xi_1\xi_2/2\xi_3 \\
&\quad - \mathrm{i}I_3\sqrt{(\xi_1\xi_2/2\xi_3)^2 + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)\xi_1\xi_2/\xi_3 - I_3^2}\,\Big] \\
&\quad \times\Big[(\xi_1\xi_2/\xi_3)\left(\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right) - I_3\right)\Big]^{-1}.
\end{aligned} \tag{4.48}$$

The canonical conjugate functions are then $-\frac{\partial S}{\partial I}(I(\xi,\gamma),\xi)$. These will still not be real functions of γ and γ^* (the complex structure is defined with respect to these coordinates).

† There is the sign ambiguity in the term $\sqrt{(\xi_1\xi_2/2\xi_3)^2 + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)\xi_1\xi_2/\xi_3 - I_3^2}$ when rewritten in the variables γ and ξ. For convenience we have chosen $\sqrt{(\xi_1\xi_2/2\xi_3)^2 + \mathrm{i}\left(\tfrac{1}{3}\hbar(2\lambda_1+\lambda_2) - \tfrac{1}{2}I_1\right)\xi_1\xi_2/\xi_3 - I_3^2} = \tfrac{1}{2}\gamma_3\xi_3 - \xi_1\xi_2/2\xi_3 - \tfrac{1}{2}\mathrm{i}\hbar\lambda_1 - \tfrac{1}{2}\gamma_2\xi_2$.

As the symplectic 2-form is real we may write it as

$$\begin{aligned}\omega_{SU_3} &= \sum_{k=1}^{3} -\mathrm{d}\frac{\partial S}{\partial I_k} \wedge \mathrm{d}I_k \\ &= \left(\sum_{k=1}^{3} -\mathrm{d}\frac{\partial S}{\partial I_k} \wedge \mathrm{d}I_k\right)^* \\ &= \sum_{k=1}^{3} -\mathrm{d}\left(\mathrm{Re}\,\frac{\partial S}{\partial I_k}\right) \wedge \mathrm{d}I_k\end{aligned} \tag{4.49}$$

and define real conjugated functions by

$$\theta_k(I,\xi) = -\mathrm{Re}\,\frac{\partial S(I,\xi)}{\partial I_k}. \tag{4.50}$$

Expressed as functions of the complex canonical coordinates γ, ξ and as functions of the non-canonical coordinates γ, γ^* they read explicitly

$$\begin{aligned}\theta_1 &= \mathrm{Re}\left(\frac{\mathrm{i}}{2}\log\frac{\xi_2}{\gamma_3}\right) = \frac{\mathrm{i}}{4}\log\left[\frac{\gamma_3^*\big(\gamma_2^*\big(1+|\gamma_1|^2\big)-\gamma_1\gamma_3^*\big)}{\gamma_3\big(\gamma_2\big(1+|\gamma_1|^2\big)-\gamma_1^*\gamma_3\big)}\right] \\ \theta_2 &= \mathrm{Re}\,(\mathrm{i}\log\xi_1) = \frac{\mathrm{i}}{2}\log\left[\frac{\lambda_1 f_2\gamma_1^* - \lambda_2 f_1\gamma_2(\gamma_3^*-\gamma_1^*\gamma_2^*)}{\lambda_1 f_2\gamma_1 - \lambda_2 f_1\gamma_2^*(\gamma_3-\gamma_1\gamma_2)}\right] \\ \theta_3 &= \mathrm{Re}\left(\mathrm{i}\log\left[\frac{2\xi_3}{\xi_1\xi_2}I_3^2 - \mathrm{i}\big(\tfrac{1}{3}\overline{h}(2\lambda_1+\lambda_2)-\tfrac{1}{2}I_1\big)\right.\right. \\ &\qquad \left.\left. +\mathrm{i}I_3\frac{2\xi_3}{\xi_1\xi_2}\sqrt{\left(\frac{\xi_1\xi_2}{2\xi_3}\right)^2 + \mathrm{i}\big(\tfrac{1}{3}\overline{h}(2\lambda_1+\lambda_2)-\tfrac{1}{2}I_1\big)\frac{\xi_1\xi_2}{\xi_3} - I_3^2}\,\right]\right).\end{aligned} \tag{4.51}$$

Canonical variables on coadjoint orbits of the SU_3 group are also given in [33, 34].

4.5. *The classical limit*

We may now perform a classical limit of a quantum-mechanical system defined on an irreducible SU_3-representation. We shall assume that the Hamiltonian H is in the enveloping algebra of $\boldsymbol{su}_3$ and that the coefficients of H scale with $\overline{h}$ in such a way that the power of $\overline{h}$ is equal to the power of the generators:

$$H = \sum_{l_1,l_2=1}^{3} a^{(1)}_{l_1l_2}\overline{h}S_{l_1l_2} + \sum_{l_1,l_2,l_3,l_4=1}^{3} a^{(2)}_{l_1l_2l_3l_4}\overline{h}^2 S_{l_1l_2}S_{l_3l_4} + \cdots \tag{4.52}$$

where the coefficients $a^{(i)}$ do not depend on $\overline{h}$. We may also allow terms which have a higher power in $\overline{h}$ than in the generators as these terms will not give any contribution in the classical limit. Performing the classical limit means of course

$$\overline{h} \to 0. \tag{4.53}$$

At the same time the density of states should increase. So at the same time the dimension of the Hilbert space has to go to infinity. As the dimension $d = \frac{1}{2}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)$ depends on two integer numbers there is some choice on the series of irreducible representations used. The most natural way seems to keep the ratio λ_1/λ_2 fixed while both λ_1 and λ_2 go to infinity. So let now λ_1, λ_2 be fixed integers and take a series $\big\{\big[\lambda_1^{(n)}, \lambda_2^{(n)}\big]\big\}_{n=1,2,3,\ldots}$ of irreducible representations

$$\lambda_1^{(n)} = n\lambda_1 \qquad \lambda_2^{(n)} = n\lambda_2 \tag{4.54}$$

where

$$\overline{h} = \frac{1}{(\lambda_1 + \lambda_2)n} \stackrel{n\to\infty}{\longrightarrow} 0. \tag{4.55}$$

The definition of a phase space, observables and a symplectic structure follows easily from the preceding sections. In fact, via the momentum map J (4.36) we have already defined classical observables $s_{ij} = \overline{h}\langle\gamma|S_{ij}|\gamma\rangle$ which are functions on a coadjoint orbit. The coadjoint orbit associated with an irreducible representation $\left[\lambda_1^{(n)}, \lambda_2^{(n)}\right]$ is uniquely given by the eigenvalues $\Lambda_{1,2}$ of the elements of the orbit (4.35). Now the whole series $\left\{\left[\lambda_1^{(n)}, \lambda_2^{(n)}\right]\right\}_{n=1,2,3,\ldots}$ is associated to one single coadjoint orbit defined by

$$\begin{aligned}\Lambda_1 &= \frac{\lambda_1^{(n)}}{\lambda_1^{(n)} + \lambda_2^{(n)}} = \frac{\lambda_1}{\lambda_1 + \lambda_2}\\ \Lambda_2 &= \frac{\lambda_2^{(n)}}{\lambda_1^{(n)} + \lambda_2^{(n)}} = \frac{\lambda_2}{\lambda_1 + \lambda_2}.\end{aligned} \tag{4.56}$$

This orbit with its symplectic structure is the classical phase space. The dimensionality of this space depends clearly on Λ_1 and Λ_2. For degenerate representations the dimension is equal to four, otherwise the classical phase space is six dimensional. The phase space and the observables s_{ij} have a clear meaning without performing the classical limit $\overline{h} \to 0$ and in fact they do not even depend on $\overline{h}$ in this limit i.e. there are no corrections to the symplectic structure and to the linear classical observables s_{ij} when $\overline{h}$ is finite. What remains is the classical limit of the dynamical equations. Thus we shall look at Heisenberg's equation and take its expectation value in coherent states

$$\left\langle\gamma\left|\frac{\mathrm{d}}{\mathrm{d}t}S_{ij}(t)\right|\gamma\right\rangle = \frac{\mathrm{i}}{\overline{h}}\langle\gamma|[H, S_{ij}(t)]|\gamma\rangle \tag{4.57}$$

where

$$S_{ij}(t) = U(t)\, S_{ij}\, U(t)^\dagger \tag{4.58}$$

and $U(t)$ is the unitary time-evolution operator. Taking expectation values in coherent states here does not mean that we make any assumption on the state of the system. Just in the spirit of the momentum map used before (4.36) this is a way to construct a function on the phase space (γ is a coordinate on the coadjoint orbit). After multiplication of (4.57) with $\overline{h}$ both sides stay finite in the limit $\overline{h} \to 0$ and we shall show that one obtains Hamilton's equations in the classical limit

$$\frac{\mathrm{d}}{\mathrm{d}t}s_{ij}(t) = \{h, s_{ij}(t)\} \tag{4.59}$$

with a Hamilton function

$$\begin{aligned}h &= \lim_{\overline{h}\to 0}\langle\gamma|H|\gamma\rangle\\ &= \sum_{l_1,l_2=1}^{3} a^{(1)}_{l_1 l_2} s_{l_1 l_2} + \sum_{l_1,l_2,l_3,l_4=1}^{3} a^{(2)}_{l_1 l_2 l_3 l_4} s_{l_1 l_2} s_{l_3 l_4} + \cdots\end{aligned} \tag{4.60}$$

(compare with (4.52)). Let us first show that the limit in the first line of (4.60) indeed gives the last line. This may be reduced to showing

$$\langle\gamma|\overline{h}S_{ij}\overline{h}S_{kl}|\gamma\rangle = s_{ij}s_{kl} + \overline{h}\partial_{kl}s_{ij} \to s_{ij}s_{kl} \qquad \text{for} \quad \overline{h} \to 0 \tag{4.61}$$

where ∂_{kl} is a differential operator which may be obtained from (3.16). Thus expectation values in coherent states factorize in the classical limit and our definition of the Hamilton function h (4.60) is correct. This factorization property may now be used to show that the commutator on the right-hand side of (4.57) in the classical limit gives the Poisson bracket in (4.59)

$$\frac{\mathrm{i}}{\hbar}\langle\gamma|[\hbar S_{ij}\hbar S_{kl},\hbar S_{rs}]|\gamma\rangle = \frac{\mathrm{i}}{\hbar}(\langle\gamma|\hbar S_{ij}[\hbar S_{kl},\hbar S_{rs}]|\gamma\rangle + \langle\gamma|[\hbar S_{ij},\hbar S_{mn}]\hbar S_{kl}|\gamma\rangle)$$
$$\xrightarrow{\hbar\to 0} s_{ij}\{s_{kl},s_{mn}\} + \{s_{ij},s_{mn}\}s_{kl} = \{s_{ij}s_{kl},s_{mn}\}. \quad (4.62)$$

In the equation above the factorization property and (4.22) are used. In the Heisenberg equation (4.57) we have used linear observables $S_{ij}(t)$—there is however no difficulty in generalizing this to products $S_{ij}S_{kl}$ (and to products of higher order).

We have not assumed anything about the state of the system in the classical limit we have defined above and indeed the limit we presented does only depend on the state of the system through initial conditions which have to be chosen appropriately. It is a very interesting question what has to be assumed for the state of the system in order to have a well defined classical limit for quantities

$$\sigma_{ij}(t) = \hbar\langle\Psi|S_{ij}(t)|\Psi\rangle \quad (4.63)$$

where $|\Psi\rangle$ is the state vector of the system (to define a classical limit we have to choose a series $\{|\Psi^{(n)}\rangle\}$ of states in some 'natural way'—each state being in a different Hilbert space). If we take a coherent state $|\Psi^{(n)}\rangle = |\gamma'\rangle$ (γ' is here a constant and not a free variable as in the definition of the functions s_{ij}) one may show that the dynamics of σ_{ij} in the limit $\hbar\to 0$ is equivalent to the classical limit defined above (any series of states with the factorization property $\langle\Psi^{(n)}|\hbar S_{ij}\hbar S_{kl}|\Psi^{(n)}\rangle \xrightarrow{n\to\infty} \sigma_{ij}\sigma_{kl}$ will have a well defined classical limit equivalent to (4.59)).

More generally one can define a sequence of statistical operators $\{\rho^{(n)}\}$ such that the Q-function

$$Q(\gamma) := \dim\mathcal{H}_{[\lambda_1,\lambda_2]}\langle\gamma|\rho^{(n)}|\gamma\rangle \quad (4.64)$$

is constant. Then $Q(\gamma)$ is a well defined probability distribution on the phase space and $\sigma_{ij}(t)$ are the time-dependent mean values of the observables $s_{ij}(t)$

$$\sigma_{ij}(t) = \int \frac{\mathrm{d}^2\gamma_1\,\mathrm{d}^2\gamma_2\,\mathrm{d}^2\gamma_3}{f_1^2 f_2^2}\,Q(\gamma)\,s_{ij}(t,\gamma). \quad (4.65)$$

5. Conclusions

In the paper we presented an explicit construction of the coherent states for the SU_3 group. We derived the relevant formulae for the expectation values of the $GL_3(\mathbb{C})$ generators. We also gave explicitly the symplectic structure on the manifolds of coherent states, which is important in derivation and investigation of the classical limit of various quantum systems with SU_3 symmetry. In our investigations we gave the results valid for an arbitrary representation of the SU_3 group, in contrast to various previous studies concerned mostly with some special representations in the context of nuclear shell models [35–38], atomic physics [23] and quantum mechanics of integrable spins [39].

Acknowledgments

We would like to thank Fritz Haake, Alan Huckleberry, Stefan Halverscheid, and Karol Życzkowski for numerous fruitful discussions. The support by SFB 237

‘Unordnung und große Fluktuationen’ and Deutsche Forschungsgemeinschaft are gratefully acknowledged.

References

[1] Klauder J R and Skagerstam B-S (eds) 1985 *Coherent States, Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[2] Schrödinger E 1926 *Naturwiss.* **14** 664
[3] Glauber R J 1963 *Phys. Rev.* **130** 2529
Glauber R J 1963 *Phys. Rev.* **131** 2766
[4] Klauder J R 1963 *J. Math. Phys.* **4** 1055
Klauder J R 1963 *J. Math. Phys.* **4** 1058
[5] Sudarshan E G G 1963 *Phys. Rev. Lett.* **10** 277
[6] Perelomov A 1972 *Commun. Math. Phys.* **26** 222
[7] Radcliffe J M 1971 *J. Phys. A: Math. Gen.* **4** 313
[8] Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* A **6** 2211
[9] Glauber R J and Haake F 1976 *Phys. Rev.* A **13** 357
[10] Yaffe L G 1982 *Rev. Mod. Phys.* **54** 407
[11] Seeger C, Kolobov M I, Kuś M and Haake F 1996 *Phys. Rev.* A **54** 4440
[12] Haake F 1991 *Quantum Signatures of Chaos* (Berlin: Springer)
[13] Gnutzmann S, Haake F and Kuś M in preparation
[14] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[15] Zhang W-M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867
[16] Puri R R 1994 *Phys. Rev.* A **50** 5309
[17] Gitman D M and Shelepin A 1993 *J. Phys. A: Math. Gen.* **26** 313
[18] Barut A O and Rączka R 1977 *Theory of Group Representations and Applications* (Warsaw: PWN-Polish Scientific)
[19] Raghunathan K, Seetharaman M and Vasan S S 1989 *J. Phys. A: Math. Gen.* **22** L1089
[20] Raghunathan K, Seetharaman M, Vasan S S and Agnes J M 1992 *J. Phys. A: Math. Gen.* **25** 1527
[21] Weigert S 1997 *J. Phys. A: Math. Gen.* **30** 8739
[22] Flanders H 1963 *Differential Forms* (New York: Academic)
[23] Chang-qi C and Haake F 1995 *Phys. Rev.* A **51** 4203
[24] Lisiecki W 1995 *Rep. Math. Phys.* **35** 327
[25] Onofri E 1975 *J. Math. Phys.* **16** 1087
[26] Berezin F A 1975 *Commun. Math. Phys.* **40** 153
[27] Woodhouse N 1980 *Geometric Quantization* (Oxford: Oxford University Press)
[28] Simon B 1980 *Commun. Math. Phys.* **71** 247
[29] Guillemin V and Sternberg S 1984 *Symplectic Techniques in Physics* (New York: Cambridge University Press)
[30] Funahashi K, Kashiwa T, Sakoda S and Fujii K 1995 *J. Math. Phys.* **36** 3232
[31] Arnold V I 1989 *Mathematical Methods of Classical Mechanics* 2nd edn (Berlin: Springer)
[32] Marsden J E and Ratiu T S 1994 *Introduction to Mechanics and Symmetry* (Berlin: Springer)
[33] Johnson K 1989 *Ann. Phys., NY* **192** 104
[34] Bulgac A and Kusnezov D 1990 *Ann. Phys., NY* **199** 187
[35] Meredith D C, Koonin S E and Zirnbauer M R 1988 *Phys. Rev.* A **37** 3499
[36] Leboeuf P and Saraceno M 1990 *J. Phys. A: Math. Gen.* **23** 1745
[37] Kurchan J, Leboeuf P and Saraceno M 1989 *Phys. Rev.* A **40** 6800
[38] Wang W, Izrailev F M and Casati G 1998 *Phys. Rev.* E **57** 323
[39] Hahn S O, Oh P and Kim M H 1996 *J. Korean Phys. Soc.* **29** 409